MA4 STOCHASTIC ANALYSIS

Xue-Mei Li (Hairer) B2.10

- Itô Integral ⟿ Stochastic Differential Equation (SDE)
  ↖ Brownian motion (BM), Martingales

stoch process:

$(\Omega, \mathcal{F}, \mathbb{P})$, State space S, complete separable metric space.
Borel σ-alg. $\mathcal{B}(S)$

Index I, $I = \mathbb{N}$, $[0,t]$ etc (time);

- A stochastic process is a family of r.v.s $(X_t, t \in I)$ on S.
  For fixed t, $X_t : \Omega \to S$
  $X : \Omega \times I \to S$
  fixed ω, $X_{(\cdot)}(\omega) : I \to S$.

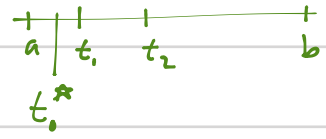
- A stochastic process $(X_t : t \in [a,b])$ is sample continuous if for almost all $\omega \in \Omega$
  $t \longmapsto X_t(\omega)$ is cts.

$\int_0^t f(B_s)\, dB_s$? ($\mathbb{R}$ valued) $\int_0^t H_t\, dB_t$? (stoch. process) "$\int_0^t H_s(\omega)\, dB_s(\omega)$" ?

1ST YR REVISION

(1) RIEMANN: A fn $f:[a,b]\to\mathbb{R}$ is Riemann integrable if there exists a number I st $\forall \varepsilon>0$ $\exists \delta>0$ st $\forall$ tagged partition $P = \{t_i : a=t_0<t_1<\cdots<t_n=b$ with $t_i^* \in [t_{i-1}, t]$ with the property that $\max_i |t_i - t_{i-1}| < \delta$, then

$$\left| \sum_1^n f(t_i^*)(t_i - t_{i-1}) - I \right| < \varepsilon$$



(2) RIEMANN-STIELTJES (RS): Let f, g be bdd fn; we say f is RS-integrable if $\exists I$ st $\forall \varepsilon>0$ $\exists \delta>0$ st if P is a tagged partition with mesh $< \delta$,

$$\left| \sum_1^n f(t_i^*)[g(t_i) - g(t_{i-1})] - I \right| < \varepsilon$$

Def A function $g:[a,b]\to\mathbb{R}$ is said to have bdd variation on $[a,b]$ if

$$\mathrm{Var}(g,[a,b]) := \sup_{P \text{ a partition}} \sum_{t_i \in P} |g(t_i) - g(t_{i-1})| < \infty$$

we write $g \in BV([a,b])$. Increasing fns & difference of increasing fns are BV. C^1 fns are BV.

Thm If $f \in C([a,b],\mathbb{R})$ & $g \in BV([a,b])$, then f is RS-integr. wrt g.

Furthermore, $\left| \int_0^t f_s\, dg_s \right| \le |f|_\infty \cdot \mathrm{Var}(g,[a,b])$

BM is not BV.

(LeL)

(3) **Thm** $\exists$ a 1-1 corresp. between a fn $g \in BV$ that is also right cts and a Radon measure on $\mathbb{R}$, given by

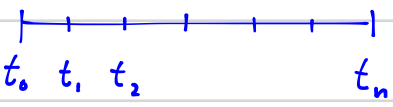
$$\mu_g([0,t]) := g(t) - g(0)$$

If f is RS-integr., then $\int_0^t f_s\, dg_s = \int_{[0,t]} f\, d\mu_g$

(4) YOUNG INTEGRAL.

I = interval

Defn A stochastic process $(X_t : t \in I)$ is said to have independent increments if for any $0 = t_0 < t_1 < \cdots < t_n$,



$\{X_{t_i} - X_{t_{i-1}}\}_1^n$ are independent r.v.s.

Defn 2.3 (BM) A real-valued stoch. process $(B_t : t \geq 0)$ is a std Brownian motion if

(1) $B_0 = 0$ a.s.

(2) B is sample cts

(3) B has independent increments

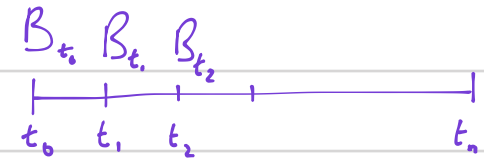
(4) $\forall\, 0 \leq s < t$, $B_t - B_s \sim N(0, t-s)$ i.e.

$$\mathbb{P}\left(\{\omega : B_t(\omega) - B_s(\omega) \leq a\}\right) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{a} e^{-x^2/2(t-s)}\, dx$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$, $f : (\Omega, \mathcal{F}) \longrightarrow (S, \mathcal{B})$ be measurable.

Then f induces a measure, $\mathbb{P} \longrightarrow f_*(\mathbb{P}) \equiv \mathbb{P} \circ f^{-1}$

given by $\forall\, A \in \mathcal{B}$, $\mathbb{P}(f^{-1}(A)) =: f_*(\mathbb{P})(A)$, the law of f

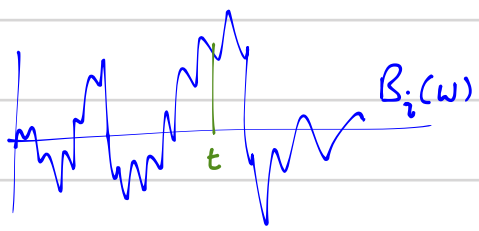


For any $t_1 < t_2 < \cdots < t_n$, $(B_{t_1}, \cdots, B_{t_n})$ induces a measure on $(S^n, \bigotimes_n \mathcal{B})$ which we call FDDs, $\mu_{t_1, \cdots t_n}$ of B_t

Ex For BM, compute the marginal distributions $\mu_{t_1, \cdots, t_n}$

BM, $B_\bullet : \Omega \longrightarrow C_0([0,T]; \mathbb{R})$ ← $C \cap \{f(0) = 0\}$ Wiener space

$\mathbb{P} \longmapsto \mu$ – Wiener measure



BM: $\Omega = C_0([0,T]; \mathbb{R})$ w/ Borel σ-alg, Wiener

$ev_t : \omega \longmapsto \omega(t)$ is BM.

Itô Integration

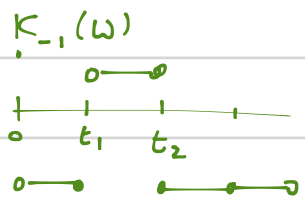
Def (1) A family of sub-σ-algs of $\mathcal{F}$, $(\mathcal{F}_t, t \in I)$ is a **Filtration** if $\mathcal{F}_s \subseteq \mathcal{F}_t$ $\forall s<t$

(2) $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is called a **Filtered probability space**

(3) A s.p. $(X_t : t \in I)$ is **adapted** (to $(\mathcal{F}_t)$) if X_t is meas. wrt $\mathcal{F}_t$ $\forall t$.

(4) A $(\mathcal{F}_t)$-BM is a BM st B is $(\mathcal{F}_t)$-adapted & $B_t - B_s$ is indep. of $\mathcal{F}_s$

Let $\mathcal{L}^{2,1}_{loc} := \left\{ (K_s) \text{ l-cts \& adapted s.p. : } \int_0^t |K_s(\omega)|^2 \, ds < \infty \text{ for a.s. } \omega \right\}$



Can define $\int_0^t K_s \, dB_s$ st

if (1) $K_t(\omega) = K_{-1}(\omega) \mathbb{1}_{\{0\}}(t) + \sum_1^\infty K_i(\omega) \mathbb{1}_{(t_{i-1}, t_i]}(t)$

$K_i \in \mathcal{F}_{t_{i-1}}$ (meas wrt t_i), then

$$\int_0^t K_s \, dB_s = \sum_1^{n-1} K_i(\omega) \left[B_{t_i}(\omega) - B_{t_{i-1}}(\omega) \right] + K_n(\omega) \left(B_t(\omega) - B_{t_n}(\omega) \right)$$

$(t_n < t < t_{n+1})$

$\int_0^t K_s \, dB_s$ is a local martingale. (Q: $\int_0^t H_s \, d\left(\int_0^t K_s \, dB_s \right)$?)

Stoch. Integral Eqn & Parabolic PDE

$\sigma_k : \mathbb{R}^d \to \mathbb{R}^d$ vector field, Borel meas., $x_0 \in \mathbb{R}^d$,

$$x_t(\omega) = x_0(\omega) + \sum_1^m \int_0^t \sigma_k(x_s) \, dB_s^k(\omega) + \int_0^t \sigma_0(x_s(\omega)) \, ds$$

a soln is an adapted s.p. st $\varphi_t(x_0, \omega)$. Assume no explosion

Markov process; $P_t f(x_0) := \mathbb{E} f(\varphi_t(x_0, \omega))$; (P_t) semigroup of op. $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_{x_i} \partial_{x_j} + \sum_1^k \sigma_0^k(x) \partial_{x_k},$$

$a_{i,j}) = \sigma \sigma^*$, $\sigma = (\sigma_1 \dots \sigma_m)$ $d \times m$ matrix; Given suitable conditions,

$P_t f$ solves $\begin{cases} \frac{\partial u_t}{\partial t} = \mathcal{L} u_t \\ u_0 = f \end{cases}$

Resources

Past lecture notes

Revuz - Yor Cts Martingales & Brownian Motion (2nd ed)

Ikeda - Watanabe SDEs & diffusion processes

Partha sarathy prob measures on metric spaces

Laplacian $\Delta (= \nabla^2)$, $f : \mathbb{R}^d \to \mathbb{R}$, $\Delta f := \sum \frac{\partial^2 f}{\partial x_i^2}$

Heat Eqn $\begin{cases} \frac{\partial u_t}{\partial t} = \frac{1}{2} \Delta u_t \\ u_0 = f \end{cases}$.

$$K_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t}, \qquad P_t(x,y) = \frac{1}{\sqrt{2\pi t}^d} e^{-|x-y|^2/2t}$$

$$= P_t(0,x) \qquad = K_t(x-y)$$

ex → $\mathbb{P}\left[x + B_t \in A\right] = \int_A P_t(x,y)\, dy$ (starting pt), $\mathbb{P}(B_t - B_s \in A) = \int_A P_{t-s}(0,y)\, dy$

$P_t(x, dy)$ Markov

Thm If $f \in L^p(\mathbb{R}^d)$ where $p \in [1, \infty]$, then

$$K_t * f(x) = \int K_t(x-y) f(y)\, dy = \int P_t(x,y) f(y)\, dy$$

Solves the HE, $\frac{\partial}{\partial t} = \frac{1}{2}\Delta$ on $(0,\infty) \times \mathbb{R}^d$.

(1) If $p < \infty$; then $K_t * f \xrightarrow[t \to 0]{L^p} f$

(2) If $f \in C^0(\mathbb{R}^d) \cap L^\infty$, then $K_t * f$ is cts on $[0,\infty) \times \mathbb{R}^d$.

(f bdd & meas)

$\mathbb{E} f(x + B_t) = \int f(y) \hat{P}_{x+B_t}(dy) = \int f(y) P_t(x,y)\, dy$

$P_t f(x)$ ← semigp

Ex $\int_A P_{t+s}(x,y)\, dy = \int_A \int_{\mathbb{R}} P_s(x,y) P_t(y,z)\, dy\, dz$

$P_t(x, A) \in \mathbb{R}$; $P_{t+s}(x, A) = \int P_s(x,y) P_t(y, A)\, dy$

B_t solves $dx_t = dB_t$

$\mathcal{G} \subseteq \mathcal{F}$, σ-algs. $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then $\exists$ a unique r.v. in $L^1(\Omega, \mathcal{G}, \mathbb{P})$ $\mathbb{E}[Y|\mathcal{G}]$ st

$$\forall A \in \mathcal{G}, \quad \int_A Y \, d\mathbb{P} = \int_A \mathbb{E}(Y|\mathcal{G}) \, d\mathbb{P}$$

Ex let $s < t$; $\mathcal{L}(B_s, B_t) = ?$

let f, g be bdd meas $\mathbb{R}$-fns on $\mathbb{R}$ ($B_t - B_s \perp\!\!\!\perp B_s$)

$$\mathbb{E}\, f(B_s)\, g(B_t) = \mathbb{E}\, f(B_s)\, g(B_s + B_t - B_s)$$

$$= \mathbb{E}\left[\mathbb{E}\left\{ f(B_s)\, g(B_s + B_t - B_s) \,\middle|\, \sigma(B_s) \right\}\right]$$

(treat B_s as fixed)

$$= \mathbb{E}\left(\int f(x)\, g(x+y)\, P_{t-s}(0, y)\, dy \,\Big|_{x = B_s} \right)$$

$$= \iint f(x)\, g(x+y)\, P_{t-s}(0,y)\, P_s(0,x)\, dy\, dx$$

$$= \iint f(x)\, g(z) \boxed{P_{t-s}(x,z)\, P_s(0,x)\, dz\, dx}$$

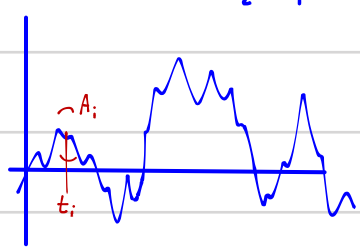
$$\Longrightarrow \mathcal{L}(B_s, B_t) = P_s(0,x)\, P_{t-s}(x,y)\, dx\, dy .$$

If $f = \chi_{A_1}$, $g = \chi_{A_2}$, $\mathbb{P}(B_s \in A_1, B_t \in A_2) = \iint_{A_1 A_2} P_s(0,x)\, P_{t-s}(x,y)\, dy\, dx$

Thm $\forall\, 0 < t_1 < t_2 < \cdots < t_n$, $A_i \in \mathcal{B}(\mathbb{R})$, $i = 1, \ldots, n$. Then

$$\mathbb{P}(B_{t_1} \in A_1, \cdots, B_{t_n} \in A_n) = \int_{A_1} \cdots \int_{A_n} P_{t_1}(0, x_1)\, P_{t_2 - t_1}(x_1, x_2) \cdots P_{t_n - t_{n-1}}(x_{n-1}, x_n)\, dx_n \ldots dx_1$$

Einstein's formulation for BM



L4 & 5

§2 Kolmogorov's Extention Theorem

$$\underbrace{P(B_{t_1} \in A_1, \cdots, B_{t_n} \in A_n)}_{\mu_{t_1 \cdots t_n}(A_1 \times A_2 \times \cdots A_n)} = \int_{A_1} \cdots \int_{A_n} p_{t_1}(0, x_1)\, p_{t_2 - t_1}(x_1, x_2) \cdots p_{t_n - t_{n-1}}(x_{n-1}, x_n)\, dx_n \cdots dx_1$$

Let $(X_\alpha, \mathcal{B}_\alpha)_{\alpha \in I}$ be measure spaces. The **cartesian product space** is

$$X_I := \prod_{\alpha \in I} X_\alpha := \left\{ x : I \to \prod_{\alpha \in I} X_\alpha \;\text{ st }\; x(\alpha) \in X_\alpha \right\}$$

Define **coordinate maps** $\pi_{\alpha_0} : \prod_{\alpha \in I} \to X_{\alpha_0}$ st $\pi_{\alpha_0}(x) = x_{\alpha_0}$ $(= ev_{\alpha_0})$

Let $\bigotimes_{\alpha \in I} \mathcal{B}_\alpha$ be the smallest σ-alg. st all π_αs are measurable.

$$\bigotimes_{\alpha \in I} \mathcal{B}_\alpha = \sigma\left\{ \pi_\alpha^{-1}(A_\alpha) \;\middle|\; A_\alpha \in \mathcal{B}_\alpha \;\; \forall \alpha \in I \right\}$$

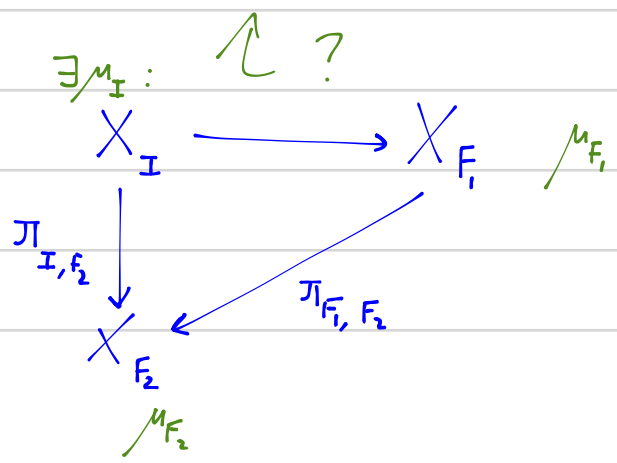
X_α = seperable complete m.space for our uses. Has ctble bases. 'standard space'.

For any $I_2 \subset I_1 \subset I$, let π_{I_1, I_2} be the **restriction/projection map** of X_{I_1} to X_{I_2} $\quad \mathbb{R}^{[0,1]} = \{x : [0,1] \to \mathbb{R}\}$ $\pi_t(x) = x(t)$, $\quad \pi_{[0,1],[0,\frac{1}{2}]}(x) = x|_{[0,\frac{1}{2}]}$

A family of measures $\{\mu_F : F \subseteq I\}$ is a **consistent** family of measures if

(1) μ_F is a measure on $\mathcal{B}_F = \bigotimes_{\alpha \in F} \mathcal{B}_\alpha$

(2) $\forall F_2 \subset F_1 \subset I$, $(\pi_{F_1, F_2})_\star \mu_{F_1} = \mu_{F_2}$



'Prob. Measures on Metric Spaces' by Parasavathy, index page

Kolmogorov's Extension Thm. Let $(X_\alpha, \mathcal{B}_\alpha)_{\alpha \in I}$ be 'standard' measure spaces, e.g. Borel sets of complete separable metric spaces.
Given a family of consistant measures $\{\mu_F, F \subset I, |F| < \infty\}$,
There exists a measure μ on X_I st $(\pi_F)_* \mu = \mu_F$. This measure is unique.

For BM, $I = [0,1]$, $X_\alpha = \mathbb{R}$, $\pi_t(x) = x(t)$.

$$\pi_t : \left(\mathbb{R}^{[0,1]}, \bigotimes_{[0,1]} \mathcal{B}(\mathbb{R}), \tilde{\mu}\right) \longrightarrow \left(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_t\right)$$

$\tilde{\mu}$ ("P")$\left[x : \pi_t(x) \in A\right] = \mu_t(A)$ by def

$$\tilde{\mu}\left(\pi_{t_i} \in A_i,\ i = 1, \cdots, n\right) = \mu_{t_1, \cdots, t_n}(A_1 \times \cdots \times A_n)$$

§3 Kolmogorov's Cty Thm.

Def Two stoch. processes $(X_t)_{t \in I}$ and $(\tilde{X}_t)_{t \in I}$ are modifications of each other if $\forall t$, $\mathbb{P}[X_t = \tilde{X}_t] = 1$.

Def X & $\tilde{X}$ are indistinguishable if $\mathbb{P}[\forall t \in I,\ X_t = \tilde{X}_t] = 1$
$\Leftrightarrow \mathbb{P}\left[\bigcup_{t \in I} X_t \neq \tilde{X}_t\right] = 0$

Thm (Kolmog.) [Revuz-Yor, Page 26, Thm 2.1]
Let $(X_t, t \in I)$ where I is an interval be a stoch. process with values in separable Banach space $(E, |\cdot|)$. $\exists$ positive constants p, δ, and c st

$$\mathbb{E}|X_t - X_s|^p \leq C|t-s|^{1+\delta}$$

Then $\exists$ a cts modification $\tilde{X}$ of X, st. $\forall \alpha \in (0, \delta/p)$,

$$\mathbb{E}\left[\sup_{\substack{t \neq s \\ t,s \in I}} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t-s|^\alpha}\right) < \infty \Rightarrow \tilde{X}_t \text{ is } \alpha\text{-Hölder cts } \mathbb{P}\text{-a.e.}$$

- 'Sobolev embedding Thm'-like
- Can use $I = [0,1]^d$ say then use $|t-s|^{d+\delta}$

Eg Let $p \geq 1$; $\mathbb{E}|B_t - B_s|^p = \frac{1}{\sqrt{2\pi}[t-s]} \int |x|^p e^{-|x|^2/2(t-s)} dx = |t-s|^{p/2} c(p)$ ($B_t - B_s \sim N(0, t-s)$)

$= |t-s|^{1+\frac{p-2}{2}}$

By the cty thm, $\exists$ a cts modification $(\tilde{B}_t)_t$ whose paths are Hölder cts of order $\alpha \in (0, \frac{p-2}{2p}) = (0, \underbrace{\frac{1}{2} - \frac{1}{p}}_{p \to \infty \to \frac{1}{2}})$ $\forall p \geq 1$.

Convention We always take the cts modification of a stoch. process if it exists.

($t \mapsto \tilde{B}_t(\omega)$ is Hölder $\alpha < \frac{1}{2}$ a.s in ω.

§4 THE WIENER MEASURE

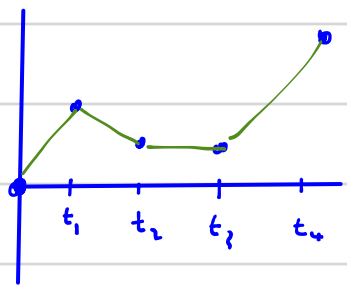
Let $W_0^d := C_0([0,1], \mathbb{R}^d)$ It is Banach under $|f|_{W_0^d} := \sup_{t \in [0,1]} |f(t)| \equiv \|f\|_\infty$.

The sets

$$\{\sigma : |\sigma(t_i) - x(t_i)| \leq r \quad \forall i \in [n]\}$$

$n \in \mathbb{N}$,

$x \in$ a ctble dense set of W_0^d



Generates $\mathcal{B}(W_0^d)$.

AIM Construct μ on $(W_0^d, \mathcal{B}(W_0^d))$ st

$$\mu(\{\sigma : \sigma_{t_1} \in A_1, \dots, \sigma_{t_N} \in A_n\}) = \mu_{t_1, \dots, t_n}(A_1 \times \dots \times A_n)$$

We construct a measurable map $\Phi : \left(\mathbb{R}^{[0,1]}, \bigotimes_{[0,1]} \mathcal{B}(\mathbb{R}^d)\right) \to (W_0^d, \mathcal{B}(W_0^d))$

$\tilde{\mu}$ —push fwd→ μ

If $x : [0,1] \to \mathbb{R}^d$ is cts when restricted to $\mathbb{Q} \cap [0,1]$,

$\Phi(x) :=$ the cts path st $\Phi(x)(t_i) = x(t_i)$ $\forall t_i \in \mathbb{Q} \cap [0,1]$

If discts $\mathbb{Q} \cap [0,1]$,

$\Phi(x) := 0$ $\forall t$. easy to check Φ meas

'K-cty Thm $\Rightarrow \tilde{\mu}(\text{cts fns}) = 1$' ?

Then $\mu := \Phi_* \mu$ is the desired Wiener measure.

Consequently, $ev_t : (W_0^d, \mathcal{B}(W_0^d), \mu) \longrightarrow \mathbb{R}$, $ev_t(\sigma) = \sigma(t)$.

Then $(ev_t, t \in [0,1])$ is a BM.

$t \longmapsto B_t(\omega)$ is not diff a.s. in ω & not of BV.

(FOR FUN SECTION)

White noise / Fourier series expansion Formally, $W(t,\omega) = "\frac{d}{dt} B_t(\omega)"$.

Fix ω, $t \longmapsto x_t(\omega)$, $x \in L^2$ say. $L^2([0,1], \mathbb{R})$

Hill. with complete o.b. $\{e_i\}_i$.

$$W(t,\omega) \overset{?}{=} \sum_i \xi_i(\omega) e_i(t), \quad \xi_i(\omega) \sim N(0,1) \text{ independent r.v.s}$$

$$x(t,\omega) := \sum_i \xi_i(\omega) \int_0^t e_i(s)\, ds \quad \text{is a BM.}$$

need to check $\mathbb{E}\left(\sum_m^{m+n} \xi_i(\omega) \int_0^t e_i(s)\, ds\right)^2 \overset{\text{INDEP}}{=} \sum_m^{m+n} \underbrace{\mathbb{E}\left(\xi_i(\omega)\right)^2}_{=1} \left(\int_0^t e_i\right)^2$

$$= \sum_m^{m+n} \left(\int_0^t e_i\right)^2 \xrightarrow[m \to \infty]{} 0$$

$\int_0^t e_i(s)\, ds = \int_0^1 \mathbb{1}_{[0,t]}(s)\, e_i(s)\, ds = \langle \mathbb{1}_{[0,t]}, e_i \rangle_{L^2}$; since $\mathbb{1}_{[0,t]} \in L^2$,

Parseval implies last step $\left(\sum (\,)^2 < \infty\right)$

— Show X_t is Gaussian process: $\mathbb{E} e^{\sum_i i\eta_i (X_{t_{i+1}} - X_{t_i})}$, $\eta_i \in \mathbb{R}$.

W_0' (Wiener measure)

$(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$: constructed each piece (if you want)

A stoch. proc $(X_t)_{t\in I}$ is L^p-integrable if $\forall t \in I$, $X_t \in L^p$ $(\Leftrightarrow \mathbb{E}|X_t|^p$

L^p-bounded if $\sup_{t\in I} \mathbb{E}|X_t|^p < \infty$

bounded if $\sup_{t,\omega} |X_t(\omega)| < \infty$ <allow for a null set>

else, do component-wise.

Def An L^1-integrable $\mathbb{R}$-valued stochastic process $(M_t : t\in I \subseteq \mathbb{R})$ adapted to $(\mathcal{F}_t)_t$ is a martingale if $\forall s,t \in I$, $s<t$,

$$\mathbb{E}[M_t | \mathcal{F}_s] = M_s \qquad \text{☆}$$

defn of $\mathbb{E}[\cdot|\mathcal{F}_s]$

$\Leftrightarrow \int_A M_t \, d\mathbb{P} = \int_A M_s \, d\mathbb{P} \quad \forall A \in \mathcal{F}_s$

$\mathbb{E}[\cdot]$

$\mathbb{E}M_t = \mathbb{E}M_s \quad \forall t<s$

notation

$\Leftrightarrow \mathbb{E}M_t \mathbb{1}_A = \mathbb{E}M_s \mathbb{1}_A \quad \forall A \in \mathcal{F}_s$

$A=\Omega$

$\Leftrightarrow \mathbb{E}(fX) = \mathbb{E}(fY) \; \forall f$ bdd meas wrt $\mathcal{F}_s$.

Eg 1 An $(\mathcal{F}_t)_t$-B.M. is an $(\mathcal{F}_t)_t$-martingale

Proof Take $s<t$: then $\mathbb{E}[B_t | \mathcal{F}_s] = \mathbb{E}[B_t - B_s + B_s | \mathcal{F}_s]$

$B_t - B_s$ is indep of $\mathcal{F}_s$; B_s meas. wrt $\mathcal{F}_s$ (adapted)

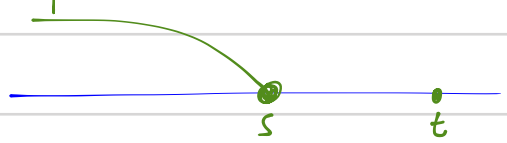
linearity $= \mathbb{E}[B_t - B_s | \mathcal{F}_s] + \mathbb{E}[B_s | \mathcal{F}_s]$

$= \mathbb{E}(B_t - B_s)^{\to 0} + B_s$ ◇

Def A Stoch. Process $(X_t)_{t\in I\subseteq\mathbb{R}}$ that is $(\mathcal{F}_t)_t$-adapted is an $(\mathcal{F}_t)_t$-Markov Process if for any $s<t$ in I, and any $f: E \to \mathbb{R}$ meas. & bdd

(E: state space)

$$\mathbb{E}[f(X_t) | \mathcal{F}_s] = \mathbb{E}[f(X_t) | \sigma(X_s)]$$



$\mathcal{F}_s^X := \sigma(X_t : t\le s) \subseteq \mathcal{F}_s$

natural filtration

Def Let $\mathcal{G} \subseteq \mathcal{F}$ and $X \in L^1(\Omega, \mathcal{F})$. Then a conditional expectation of X given $\mathcal{G}$ is any $\mathcal{G}$-measurable r.v. $Y \in L^1(\Omega, \mathcal{G})$

(#) ——— $\int_A X \, d\mathbb{P} = \int_A Y \, d\mathbb{P} \quad \forall A \in \mathcal{G}$.

Thm $\forall X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ $\exists$ $Y \in L^1(\Omega, \mathcal{G}, \mathbb{P})$ satisfying (#) ($\mathcal{G} \leq \mathcal{F}$)
and if $Y_1, Y_2 \in L^1(\Omega, \mathcal{G}, \mathbb{P})$ satisfying (#), then $Y_1 = Y_2$ $\mathbb{P}$-a.s.
We write $\mathbb{E}[X \mid \mathcal{G}]$ the cond. exp. of X on $\mathcal{G}$.

Proof ① "!" Let $Y_1, Y_2 \in L^1(\Omega, \mathcal{G}, \mathbb{P})$ st (#) holds. Then $Y_1 - Y_2 \in L^1(\mathcal{G})$
and $\forall A \in \mathcal{G}$, $\int_A Y_1 - Y_2 \, d\mathbb{P} = 0 \Rightarrow Y_1 - Y_2 = 0$ a.s. so $Y_1 = Y_2$ a.s.

② If $a, b \in \mathbb{R}$, $X_1, X_2 \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, then
$\mathbb{E}[aX_1 + bX_2 \mid \mathcal{G}] = a\,\mathbb{E}[X_1 \mid \mathcal{G}] + b\,\mathbb{E}[X_2 \mid \mathcal{G}]$ a.s.
↙ check it satisfies (#) with $X = aX_1 + bX_2$.

③ $X \geq 0$, $A \in \mathcal{G}$, define $\mathbb{Q}(A) := \int_A X \, d\mathbb{P}$
Then $\mathbb{Q} \geq 0$ & σ-additive, $\mathbb{P}$ is prob. meas. } $\mathbb{Q}$ is a measure s.t.

$\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0$ $\forall A \in \mathcal{G}$ $\overset{\text{defn}}{\iff}$ $\mathbb{Q} \ll \mathbb{P}$, $\mathbb{Q}$ absolutely continuous wrt $\mathbb{P}$

By Radon-Nikodym Thm, there exists a $\mathcal{G}$-meas. fn $\frac{d\mathbb{Q}}{d\mathbb{P}}$ st

$\mathbb{Q}(A) = \int_A \frac{d\mathbb{Q}}{d\mathbb{P}} \, dx$

eg. $\mathbb{P} = dx$, $\mathbb{Q} = f(x)\,dx$
$\int g \, d\mathbb{Q} = \int g(x) f(x) \, dx$
$\mathbb{Q} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, \ldots$

So $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is the conditional expectation (a.s.)

④ If $X = X^+ - X^-$, $X^\pm \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{E}[X \mid \mathcal{G}] := \mathbb{E}[X^+ \mid \mathcal{G}] - \mathbb{E}[X^- \mid \mathcal{G}]$

L 7, 8

Properties of

$$\int_A \mathbb{E}[X|\mathcal{G}]\,d\mathbb{P} = \int_A X\,d\mathbb{P} \quad \forall A \in \mathcal{G}.$$

(0) linearity

(1) If $X \geq 0$, $\mathbb{E}[X|\mathcal{G}] \geq 0$ ← This is clear since $\mathbb{E}[X|\mathcal{G}]$ is the

(2) If $X \geq Y$, $\mathbb{E}[X|\mathcal{G}] \geq \mathbb{E}[Y|\mathcal{G}]$ R-N-deriv $\frac{d\mathbb{Q}}{d\mathbb{P}}$, $\mathbb{Q}(A) := \int_A X\,d\mathbb{P}$ $(A \in \mathcal{G})$

(3) $\mathbb{E}(\mathbb{E}[X|\mathcal{G}]) = \mathbb{E}X$ Alt pf: use $A = \{\mathbb{E}[X|\mathcal{G}] < 0\}$

(4) L^1 contractivity $\mathbb{E}\left|\mathbb{E}[X|\mathcal{G}]\right| \leq \mathbb{E}|X|$

pf let $X = X^+ - X^-$; $\left|\mathbb{E}[X|\mathcal{G}]\right| = \left|\mathbb{E}[X^+|\mathcal{G}] - \mathbb{E}[X^-|\mathcal{G}]\right| \leq \mathbb{E}[X^+|\mathcal{G}] + \mathbb{E}[X^-|\mathcal{G}]$

$= \mathbb{E}[|X||\mathcal{G}]$

— MCT:

If (f_n) are non-decreasing seq. of rvs s.t. $f_n \uparrow f$ a.s.

and $\exists$ an integrable fn g st $f_n^- \leq g$. Then $\lim_{n\to\infty} \int f_n\,d\mathbb{P} = \int \lim_{n\to\infty} f_n\,d\mathbb{P}$

Disintegration & Orthogonal Projection

2nd pf for "$\exists$!" of cond. exp.

Let $\mathcal{G} \leq \mathcal{F}$ as σ-algebras.

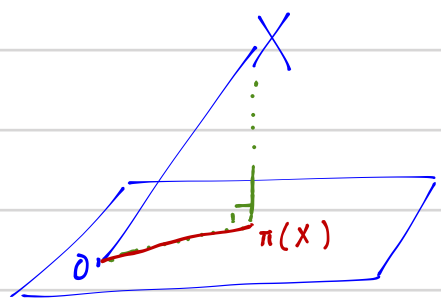
Then $L^2(\Omega, \mathcal{G}, \mathbb{P}) \subseteq L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Recall $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Hilbert space with $\langle X, Y\rangle_{L^2} := \int_\Omega XY\,d\mathbb{P}$

And $|X|_{L^2}^2 := \int_\Omega |X|^2\,d\mathbb{P}$ polarise $\rightsquigarrow \langle\cdot,\cdot\rangle_{L^2}$

It is easy to see (ex) $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a closed subspace.

Let $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Let $\pi(X)$ be the orthogonal projection of f to $L^2(\Omega, \mathcal{G}, \mathbb{P})$. Then $\pi(X) = \mathbb{E}[X|\mathcal{G}]$.



To see this, note

$$X - \pi(X) \perp L^2(\Omega, \mathcal{G}, \mathbb{P})$$

In particular, $\forall A \in \mathcal{G}$, $\mathbb{1}_A \in L^2(\Omega, \mathcal{G}, \mathbb{P})$

$$\Rightarrow \langle X - \pi(X), \mathbb{1}_A\rangle = 0 \iff \int_A X\,d\mathbb{P} = \int_A \pi(X)\,d\mathbb{P}.$$

So $\pi(X) = \mathbb{E}[X|\mathcal{G}]$.

extend to L^1:

Let $X \geq 0$ be in L^1. Let $X_n = X \mathbb{1}_{|X| < n}$:

Then (X_n) is a seq. of ≥ 0 bdd fns non-decreasing in n st $X_n \uparrow X$.
Then $\pi(X)$ increases & is ≥ 0. (properties ① & ②).

$$\int_A \lim_{n \to \infty} \pi(X_n) \mathbb{1}_A \quad \text{exists} \quad \forall A \in \mathcal{G}$$

we prove $\lim \pi(X_n) = \mathbb{E}[X|\mathcal{G}]$: By MCT,

$$\lim_{n \to \infty} \int_A \pi(X_n)\, d\mathbb{P} = \int_A \lim_{n \to \infty} \pi(X)\, d\mathbb{P}$$

But also, $\lim_{n \to \infty} \int_A X_n\, d\mathbb{P} = \int_A X\, d\mathbb{P} < \infty$ (DCT)
($X \in L^1$)

$$\Rightarrow \int_A \lim_{n \to \infty} \pi(X_n)\, d\mathbb{P} = \int_A X\, d\mathbb{P}. \Rightarrow \lim \pi(X_n) \in L^1 \text{ \& is the cond. exp.}$$

Finally if $X \in L^1$, let $X = X^+ - X^-$, then $\mathbb{E}[X|\mathcal{G}] := \mathbb{E}[X^+|\mathcal{G}] - \mathbb{E}[X^-|\mathcal{G}]$ is the cond. exp.

Prop Let $X \in L^1$.

① If $\mathcal{G}_1 \leq \mathcal{G}_2 \leq \mathcal{F}$ are σ-algebras, then

$$\mathbb{E}\left[\mathbb{E}(X|\mathcal{G}_1) \,\middle|\, \mathcal{G}_2\right] = \mathbb{E}\left[\mathbb{E}(X|\mathcal{G}_2) \,\middle|\, \mathcal{G}_1\right] = \mathbb{E}[X|\mathcal{G}_1]$$

② If $Y \in \mathcal{G}$ st $XY \in L^1$, $\mathbb{E}[XY|\mathcal{G}] = Y\,\mathbb{E}[X|\mathcal{G}]$ check defn works

③ [Jensen] Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. Then

$$\phi\left(\mathbb{E}[X|\mathcal{G}]\right) \leq \mathbb{E}\left[\phi(X) \,\middle|\, \mathcal{G}\right].$$

③ eg. $\phi(x) = |x|^p$, e^x.

④ [cDCT] If $|X_n| \le g \in L^1$ $\forall n$, $X_n \to X$ a.s., then

$$\mathbb{E}[X_n|\mathcal{G}] \to \mathbb{E}[X|\mathcal{G}]$$

⑤ [cMCT] If $X_n \ge 0$ $\forall n$, $X_n \uparrow X$ a.s. then

$$\mathbb{E}[X_n|\mathcal{G}] \nearrow \mathbb{E}[X|\mathcal{G}].$$

⑥ [cFATOU] If $X_n \ge 0$ then $\mathbb{E}(\liminf_{n\to\infty} X_n|\mathcal{G}) \le \liminf_{n\to\infty} \mathbb{E}[X_n|\mathcal{G}]$.

10 MIN BREAK L8

Notation.

$$\mathbb{E}[Y|\sigma(\{X_i\}_{i=1}^{\infty})] =: \mathbb{E}[Y|\{X_i\}_1^{\infty}]$$

$$\mathbb{P}(B|\mathcal{G}) := \mathbb{E}[1_B|\mathcal{G}] \quad \text{a.s.} \ (\text{not necessarily a measure})$$

(can be Banach-valued)

Thm Let X, Y be $\mathbb{R}$-valued r.v.s. If Y is $\sigma(X)$-meas, then $\exists!$ a Borel meas fn $f: \mathbb{R} \to \mathbb{R}$ s.t. $Y = f(X)$.

Notation Let $x \mapsto \mathbb{E}[Y|X=x]$ be the borel meas fn g s.t. $\mathbb{E}[Y|X] = g(X)$

Let $\hat{\mathbb{P}}_X = (X_*)\mathbb{P}$ be the dist. of X. Let $B \in \mathcal{B}(\mathbb{R})$. Then

$$\int_B \underbrace{\mathbb{E}(Y|X=x)}_{g(x)} \, d\hat{\mathbb{P}}_X(x) = \int_{X^{-1}(B)} g(X) \, d\mathbb{P}$$

$$= \int_{X^{-1}(B)} \mathbb{E}[Y|X] \, d\mathbb{P}$$

$$= \int_{X^{-1}(B)} Y \, d\mathbb{P}$$

Example (X,Y) has joint distrib $\rho(x,y)\,dx\,dy$ for some $\mathcal{B}$-meas. ρ.

what is $\hat{P}_X(dx)$? $\mathbb{P}(X\in A,\ Y\in\mathbb{R})$

$$\hat{P}_X(dx) = \left(\int_{\mathbb{R}} \rho(x,y)\,dy\right)dx \qquad \langle \text{exercise} \rangle$$

$$\int_B \mathbb{E}[Y\mid X=x]\left(\int_B \rho(x,y)\,dy\right)dx = \int_{X^{-1}(B)\leftarrow\{X\in B\}} Y\,d\mathbb{P} = \iint_{\mathbb{R}^2} \mathbb{1}_B(x)\,y\,\rho(x,y)\,dx\,dy.$$

$$= \int_B \int_{\mathbb{R}} y\,\rho(x,y)\,dy\,dx$$

$$\Longrightarrow \quad \mathbb{E}[Y\mid X=x] = \frac{\int_{\mathbb{R}} y\,\rho(x,y)\,dy}{\int_{\mathbb{R}} \rho(x,y)\,dy}$$

$$\Longrightarrow \quad \mathbb{E}[Y\mid X](\omega) = \frac{\int_{\mathbb{R}} y\,\rho(X(\omega),y)\,dy}{\int_{\mathbb{R}} \rho(X(\omega),y)\,dy} \quad \text{a.s.}$$

Exercise compute $\mathbb{E}[B_t\mid B_1]$, $\mathbb{P}(B_{t_i}\in A_i,\ i\in[n]\mid B_1)$

(eg) $\exists (A_i)_1^N$ disjoint s.t. $\bigsqcup A_i = \Omega$. $\mathcal{G} = \sigma(A_1, A_2, \dots, A_n)$,
so $A_1,\dots,A_n$ are atom sets for $\mathcal{G}$, $\mathbb{P}(A_i)>0$

If $X\in L^1$, $\mathbb{E}[X\mid\mathcal{G}] = ?$

Guess $\int_{A_i} X\,d\mathbb{P} = \int_{A_i} \sum_{i=1}^{n} \frac{\mathbb{E}[X\mathbb{1}_{A_i}]}{\mathbb{P}(A_i)}\mathbb{1}_{A_i}\,d\mathbb{P}$

Exercise: check $\int_A \mathbb{E}[X\mid\mathcal{G}] = \int_A X\,d\mathbb{P}$.

$\mathbb{E}[X|\mathcal{G}](\omega)$ is ① projection

② $\int X(\omega')\, d[\mathbb{P}(\omega'|\mathcal{G})(\omega)]$ if it exists!

Emphasise If $X \in L^2$, $\mathbb{E}[X|\mathcal{G}]$ is the $\mathcal{G}$-meas L^2 r.v. that minimises $\mathbb{E}|X-Y|^2$ among $Y \in L^2(\Omega, \mathcal{G}, \mathbb{P})$

i.e. $\mathbb{E}|X - \mathbb{E}[X|\mathcal{G}]|^2 = \inf_{Y \in L^2(\Omega,\mathcal{G},\mathbb{P})} \mathbb{E}|X-Y|^2$

UNIFORM INTEGRABILITY

Def A family of $\mathbb{R}$-valued fns on a meas space (Ω, Σ, μ), $(f_\alpha : \alpha \in \Lambda)$ is uniformly integrable if

$\forall \varepsilon > 0$, $\exists C_0 > 0$ st if $C > C_0$,

$$\int_{\{|f_\alpha| \geq C\}} |f_\alpha|\, d\mu \leq \varepsilon$$

$$\Leftrightarrow \lim_{C \to \infty} \sup_{\alpha \in \Lambda} \int_{\{|f_\alpha| \geq C\}} |f_\alpha|\, d\mathbb{P} = 0$$

Thm If $X_n \xrightarrow{\mathbb{P}} X$. Let $p \geq 1$. Then $X_n \xrightarrow{L^p} X$ iff $\{|X_n|^p : n \in \mathbb{N}\}$ is U.I.

L^p-bdd, $p > 1 \Longrightarrow$ UI

Thm (Vitali) For $p\in[1,\infty)$. Let $f_n \in L^p(\mu)$, $n\in\mathbb{N}$. TFAE

① $f_n \longrightarrow f$ in L^p.

② $\{|f_n|^p : n\in\mathbb{N}\}$ is U.I. & $f_n \longrightarrow f$ in measure.

$\Longrightarrow f_n \to f$ in measure, $\int f_n \to \int f$

③ $\int |f_n|^p \, d\mu \longrightarrow \int |f|^p \, d\mu$ & $f_n \longrightarrow f$ in measure.

($\sup_n \|f_n\|_p < M$).

Thm If (f_n) is L^p-bdd, $p\geq 1$, then $\{f_n\}$ is UI.

Pf Markov Ineq.

$$\int_{|f_\alpha|>c} |f_\alpha| \, d\mu \leq \int \frac{|f_\alpha|^p}{c^{p-1}} \, d\mu \leq \frac{1}{c^{p-1}} \sup_\alpha \|f_\alpha\|_p \longrightarrow 0 \quad (c\to\infty)$$

□

Ch 2 Martingale Theory

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$ be a FPS. $I=[a,b]$, $[a,\infty)$, $\mathbb{N}$, ...

Std assumptions: ① $\mathcal{F}_t$ contains all ($\mathbb{P}$-)null sets from $\mathcal{F}_\infty := \bigvee_{t\in I} \mathcal{F}_t$

② $\mathcal{F}_{t+} := \bigcap_{h>0} \mathcal{F}_{t+h}$; A filtration is right-cts if $\mathcal{F}_{t+} = \mathcal{F}_t$.

we assume $\mathcal{F}_t = \mathcal{F}_{t+}$.

Let $\hat{\mathcal{F}}_t := \mathcal{F}_{t+}$; then $(\hat{\mathcal{F}}_t)_{t\in I}$ is right cts. $\hat{\mathcal{F}}_t :=$ augmented σ-alg.

Remark The natural filtration of a brownian motion after completion is right cts. (Blumenta's 0-1 law)

$\forall t,\ X_t \in L^1$

Defn 2.1 An $(\mathcal{F}_t)$-adapted L^1 stoch. process $(X_t)_{t\in I}$ is

(1) A martingale if $\forall s<t \in I$,
$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s \quad \left(\text{OR } \mathbb{E}X_t \mathbb{1}_A = \mathbb{E}X_s \mathbb{1}_A \ \forall A \in \mathcal{F}_s\right)$$

(2) A super-martingale if $\forall s<t \in I$,
$$\mathbb{E}[X_t \mid \mathcal{F}_s] \le X_s$$

(3) A sub-martingale if $\forall s<t \in I$,
$$\mathbb{E}[X_t \mid \mathcal{F}_s] \ge X_s$$

Eg 2.1. Let $f \in L^1$. Define $X_t := \mathbb{E}(f \mid \mathcal{F}_t)$. Then X_t is a mgale.

Pf Let $s<t$, $\mathbb{E}[X_t \mid \mathcal{F}_s) = \mathbb{E}[(\mathbb{E}f \mid X_t) \mid X_s] = \mathbb{E}(f \mid X_s) = X_s$.

ex $(X_t)_{t\in I}$ is UI. converse later

§2 Discrete Martingales

Prop 2.1 $(X_n)_n$ is a martingale iff $\forall n$, $\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] = X_{n-1}$.

Pf easy. □

(unbiased)

Eg 2.2 (1) (Random Walk) Let $(X_i)_i$ be independent r.v.s with mean 0.
Define $S_n = \sum_1^n X_i$, $\mathcal{F}_n = \sigma\{X_1, \dots, X_n\}$.

Then S_n is a martingale.

Proof $\mathbb{E}[S_n \mid \mathcal{F}_{n-1}] = \mathbb{E}[X_n + \Sigma_1^{n-1} X_n \mid \mathcal{F}_{n-1}]$ ($X_n \perp\!\!\!\perp \mathcal{F}_{n-1}$) $= \mathbb{E}X_n + \Sigma_1^{n-1} X_n$
$= S_{n-1}$ □

(2) Let $Y_i = X_i + 1$ (so $\mathbb{E}Y_i = 1$), $\tilde{S}_n = \sum_1^n Y_i = n + \sum_1^n X_i$.
Then $\tilde{S}_n$ is a sub-martingale. (ex).

(3) $\{Y_n\}$ — indep r.v.s, $\mathbb{E}Y_n = 1$. Let $M_n = \prod_1^n Y_i$. Then M_n is a $\sigma((Y_i)_i)$-martingale

Wed W4
L10~11

A fn $F:[a,b]\to\mathbb{R}$ is abs. cts iff $F(x)=F(a)+\int_a^x f(t)\,dt$, $f\in L^1$.
$f\in L^1 \Leftrightarrow f^+$ & $f^-\in L^1$

X_t submartingale (Revuz-Yor: integrable sub-mart $\equiv$ our sub-mart.)

§2 Discrete Integral.

Let $(X_n)_{n\in\mathbb{N}}$ be $(\mathcal{F}_n)_n$-adapted. Let $H_n\in\mathcal{F}_{n-1}$ (previsible). The total winning at time n is (amt you gamble.)

$$(H\cdot X)_n = H_1(X_1-X_0)+H_2(X_2-X_1)+\cdots+H_n(X_n-X_{n-1})$$

The process $(H\cdot X)_n =: \int_0^n H_s\,dX_s$ is the discrete integral or the martingale transform of X by H.

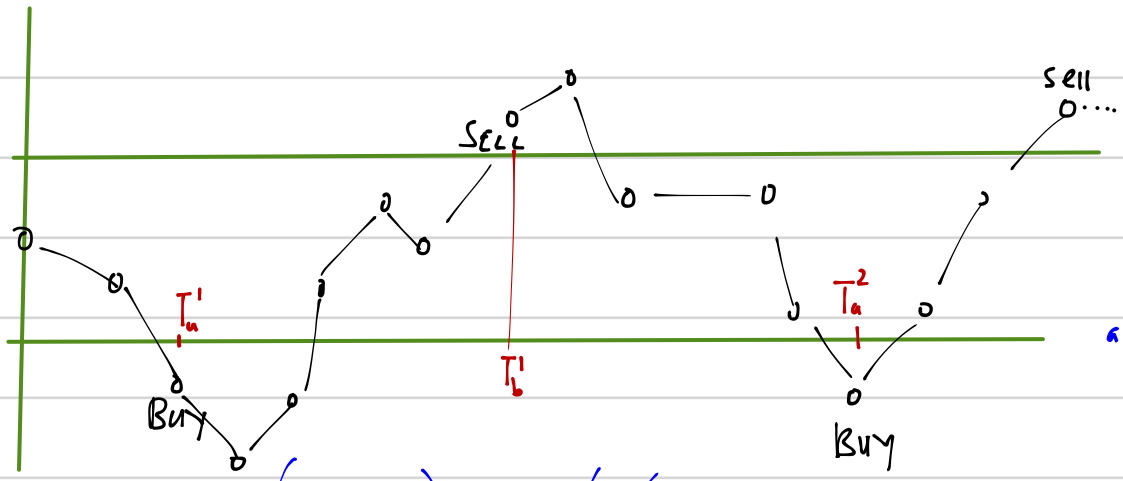
Thm 2.2 (Strategy (H_n) does not change the game)
Let $H_n\in\mathcal{F}_{n-1}$ with $|H_n(\omega)|\le K$ for some number K and all $\omega\in\Omega$.
stock going down → Fair ⇒ Fair
① If $H_n\ge 0$ & (X_n) is a super-mgale, then so is $H\cdot X$.
② If X_n is a martingale, then so is $H\cdot X$

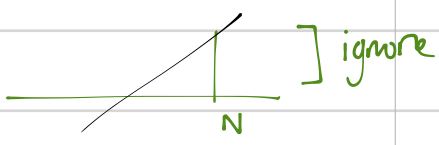
Proof $\mathbb{E}[(H\cdot X)_{n+1}\mid\mathcal{F}_n] = (H\cdot X)_n + \mathbb{E}[H_{n+1}(X_{n+1}-X_n)\mid\mathcal{F}_n]$

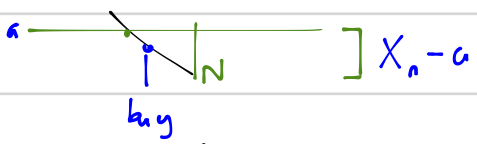
$$= \begin{cases} = (H\cdot X)_n & \text{if } X \text{ is a mart.} \\ \le (H\cdot X)_n & \text{if } X \text{ is a supmart.} \end{cases}$$

This prove ① & ②.
note $(H\cdot X)_n$ is integrable since each summand is.

UPCROSSING THM







sell at N

$$(H \cdot X)_N \geq (b-a) U_N([a,b]) - (X_N - a)^-$$

total # of upcrossings from a to b by time N

negative part

$\star$

Thm 2.3 [Doob's Upcrossing Thm.] Let X_n is a supermart. Then

$$(b-a)\,\mathbb{E}\,U_n([a,b]) \leq \mathbb{E}(X_N - a)^-$$

Pf Since X_n is a supermart, $(H \cdot X)_n$ is a supermart when H_n is the strategy

$$\begin{cases} 1 & \text{if } X_n \leq a \\ 0 & \text{if } X_n \geq b \\ \text{and hold the value} & \text{at intermediate times} \end{cases}$$

By previous Thm (2.2), $\mathbb{E}(H \cdot X)_N \leq \mathbb{E}(H \cdot X)_0 = 0$.

Taking expectation of $\star$ to conclude. $\square$

Thm 2.4 [Mgale Convergence Thm I]

Let X_n be a supermart with $\sup_n \mathbb{E} X_n^- < \infty$.
Then $\lim_{n\to\infty} X_n$ exists.
If furthermore $\sup_n \mathbb{E}|X_n| < \infty$ then $\lim X_n \in L^1$.

Pf $\lim X_n(\omega)$ exists iff $\overline{\lim} X_n(\omega) = \underline{\lim} X_n(\omega)$.

$\Longrightarrow \exists\ a < b$ st $\underline{\lim} X_n(\omega) \le a < b \le \overline{\lim} X_n(\omega)$.

Consider $\Lambda_{a,b} := \{\omega : \underline{\lim} X_n(\omega) \le a < b \le \overline{\lim} X_n(\omega)\}$

Then $\{\omega : \lim X_n(\omega) \text{ DNE}\} \subset \bigcup_{a,b\in\mathbb{Q}} \Lambda_{a,b}$

Doob $\Rightarrow$ $\mathbb{E} U_n([a,b]) \le \frac{1}{b-a}\mathbb{E}(X_n - a)^- \le \frac{1}{b-a}\mathbb{E}(X_n^- + |a|)$.

MCT $\Longrightarrow$ $\mathbb{E}\lim_{n\to\infty} U_n([a,b]) = \lim_{n\to\infty} \mathbb{E} U_n([a,b]) \le \frac{1}{b-a}\sup_n \mathbb{E}[X_n^- + |a|]$.
$< \infty$ by assumption.

Hence $\lim_{n\to\infty} U_n([a,b]) < \infty$ a.s. for any $a < b$.

$\Longrightarrow P(\Lambda_{a,b}) = 0$.

$\Rightarrow \lim_{n\to\infty} X_n(\omega)$ exists a.s..

If $\sup_n \mathbb{E} X_n < \infty$, then $\mathbb{E}\lim_{n\to\infty}|X_n| \le \underline{\lim}_{n\to\infty} \mathbb{E}|X_n| < \sup_n \mathbb{E}|X_n| < \infty$. (Fatou)

□

Cádlag = Continue à droite et limit a gauche
= r-cts & L-limit.

Rem $t \in [0, T)$, $T = \infty$ or finite.

$\S$ $(\mathcal{F}_t)_t$ is R-cts & complete.
Let $Y \in L^1$. We may choose Y_t from equivalence class of $\mathbb{E}[Y|\mathcal{F}_t]$ st (Y_t) is a cádlag martingale.

(☆) **Cor** (1) Let X_t be R-cts adapted to $\mathcal{F}_t$. Then if X_t is a supermart & $\sup_{t \in [0,T)} \mathbb{E} X_t^+ < \infty$. Then $\lim_{t \to T} X_t$ exists a.s.

(2) If (X_t) is a submart & $\sup_{[0,T)} \mathbb{E} X_t^- < \infty$ then $\lim_{t \to T} X_t$ exists a.s.

Rem a +ve supermart $(X_t : t < \infty)$ always has a limit $X_\infty := \lim_{t \to \infty} X_t$.

STOPPING TIMES
$I = [0, \infty)$ or $I = \mathbb{N}$ $\quad (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, \mathbb{P})$

Def 2.2 (1) Let $T : \Omega \to I \cup \{\infty\}$, T is a stopping time if $\forall\, t \in I$,

$$\{\omega : T(\omega) \le t\} \in \mathcal{F}_t$$

(2) If X_t is a stoch. process, T a s.t., we define X^T to be the process stopped at T,

$$X_t^T(\omega) = X_{T(\omega) \wedge t}(\omega) = \begin{cases} X_t(\omega) & t \le T(\omega) \\ X_{T(\omega)}(\omega) & t > T(\omega) \end{cases}$$

(3) Let T be a s.t.. Then

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap \{T \le t\} \in \mathcal{F}_t, \forall\, t\}.$$

Rem Let S, T be s.t.s ① If $S \le T$ then $\mathcal{F}_S \subseteq \mathcal{F}_T$.
② If $S \le T$, $A \in \mathcal{F}_S$, $S\mathbb{1}_A + T\mathbb{1}_{A^c}$ is a s.t. (ex)

Prop 2.6 Let T be a s.t.
① If (X_n) is a supermart, so is X^T.
② If (X_n) is a mart, so is X^T.

Pf $X_n^T = \sum_1^{n-1} X_i \mathbb{1}_{\{T=i\}} + X_n \mathbb{1}_{\{T \ge n\}}$

Then $\mathbb{E}|X_n^T| < \infty \quad \forall n$.

$$\{T \ge n\} = \Omega \setminus \{T \le n-1\} \in \mathcal{F}_{n-1}$$

Now $\mathbb{E}[X_n^T | \mathcal{F}_{n-1}] = \sum_1^{n-1} X_i \mathbb{1}_{T=i} + \underbrace{\mathbb{E}[X_n | \mathcal{F}_{n-1}]}_{= X_{n-1} \text{ if mart}} \mathbb{1}_{T \ge n} = \begin{cases} X_{n-1}^T & \text{mart} \\ \le X_{n-1}^T & \text{supermart.} \end{cases}$
$\le X_{n-1}$ if supermart

□

$\{T \le n\} \in \mathcal{F}_n \ \forall n$

Prop 2.5 $T: \Omega \to \mathbb{N} \cup \{\infty\}$ is a stopping time iff $\{T=n\} \in \mathcal{F}_n \ \forall n$.

Pf ⇐ If $[T=n] \in \mathcal{F}_n \ \forall n$, $[T \le n] = \bigcup_1^n \underbrace{[T=i]}_{\in \mathcal{F}_i \subseteq \mathcal{F}_n} \in \mathcal{F}_n \Rightarrow T$ is s.t.

⇒ If T is s.t., $[T=n] = [T \le n] \setminus [T \le n-1] \in \mathcal{F}_n$.

Eg 2.2 Let (X_n) be an adapted process. Let $T_1 = \inf_{n \ge 1} \{X_n \ge a\}$,

T_i are s.t.s.

$T_2 = \ \le$
$T_3 = \ >$

$[T_1 = n] = [X_1 < a] \cap [X_2 < a] \cap \cdots \cap [X_n \ge a] \in \mathcal{F}_n$.

(ex) check T_2, T_3.

L

$\mathbb{E}[X_n | \mathcal{F}_{n-1}] \le X_{n-1}$

PROP 2.7 (Doob's elementary Optional Stopping Thm)

Let (X_n) be a supermart & let $S \le T$ be bounded s.t.s, $S \le T \le N$.
Then $\mathbb{E}X_T \le \mathbb{E}X_S$.
If furthermore X is a mart, then $\mathbb{E}X_T = \mathbb{E}X_S$.

Remark: (1) only need $(X_i : i \in I)$, I ctble & ordered.
(2) $\mathbb{E}X_T \le \mathbb{E}X_S$ $\overset{\text{we will see}}{\iff}$ $\mathbb{E}[X_T | \mathcal{F}_S] \le X_S$ $\forall\ S \le T \le N$.

PROOF (1) Let $H_n := \mathbb{1}_{T \ge n} - \mathbb{1}_{S \ge n} :=$

1	$T \ge n$	$S < n$
0	$T \ge n$	$S \ge n$
	or	
	$T < n$	$S < n$
-1	$T < n$	$S \ge n$

$$(H \cdot X)_N = \sum_{S \le i \le T} (X_{i+1} - X_i) = X_T - X_S$$

Since $H \cdot X$ is a supermart.:

$\mathbb{E}(H \cdot X)_N \le \mathbb{E}(H \cdot X)_0 = 0$ "=" if martingale
$\Rightarrow \mathbb{E}X_T - \mathbb{E}X_S \le 0$ □

(v2) Let $A \in \mathcal{F}_S$. Then $A \cap \underbrace{[S = r]}_{=:} \in \mathcal{F}_r$, $\forall r \in I$.

just for martingale. $X_S \mathbb{1}_{A \cap A_r} = X_r \mathbb{1}_{A \cap A_r}$

$\Rightarrow \mathbb{E}X_S \mathbb{1}_{A \cap A_r} = \mathbb{E}X_r \mathbb{1}_{A \cap A_r} \underset{(=)}{\ge} \mathbb{E}(X_N \mathbb{1}_{A \cap A_r})$

Since $X_N \in L^1$, summing gives

$\mathbb{E}X_S \mathbb{1}_A \underset{(=)}{\ge} \mathbb{E}X_N \mathbb{1}_A \quad \forall A \in \mathcal{F}_S$

$\Rightarrow \mathbb{E}[X_n | \mathcal{F}_S] \le X_S \overset{\mathbb{E}[\cdot]}{\Rightarrow} \mathbb{E}X_N \underset{(=)}{\le} \mathbb{E}X_S$.

$\left(\tau = S\mathbb{1}_A + T\mathbb{1}_{A^c}\right.$ for supermart$\left.\right)$.

Thm 2.12 Let $(X_t : t \in I)$ be integrable stoch. process [with I ctble or I is an interval $([0,T)$ or $[0,\infty)$). In latter case, also assume X is R-cts.] adapted to $\mathcal{F}_t$.
Let S, T be s.t.s. Then:

(1) If $\forall$ bdd $S \le T$, $\mathbb{E}X_S = \mathbb{E}X_T$, then $\mathbb{E}[X_T | \mathcal{F}_S] = X_S$ $\forall$ bdd $S \le T$.

(2) ——"—— $\mathbb{E}X_S \le \mathbb{E}X_T$ Then $\mathbb{E}[X_T | \mathcal{F}_S] \ge X_S$ ——"——.

Pf Let $S \le T \le K$, $K \in \mathbb{R}_+$. Let $A \in \mathcal{F}_S$, $\tau := S\mathbb{1}_A + T\mathbb{1}_{A^c}$.
Then $\tau \le k$ is a s.t. (remark after prop 2.5)

(1) $\mathbb{E}X_\tau = \mathbb{E}X_T$

$$\iff \mathbb{E}[X_S \mathbb{1}_A + X_T \mathbb{1}_{A^c}] = \mathbb{E}[X_T \mathbb{1}_A + X_T \mathbb{1}_{A^c}]$$

$$\iff \mathbb{E}X_S \mathbb{1}_A = \mathbb{E}X_T \mathbb{1}_A.$$

(2) Ex. □

Ex $\nexists$ $X_\infty := \lim_{t\to\infty} X_t \in L^1$. Prove an analogous statement. HINT: ∞ is s.t.

Cor (X_t) is a martingale iff $\forall$ bdd sts T, $\mathbb{E}X_T = \mathbb{E}X_0$.

Prop (Doob EOST v2 (Real version.)) Let S, T be s.t.s, $S \le T \le N$.
Then if (X_n) is a supermart.

$$\mathbb{E}[X_T | \mathcal{F}_S] \le X_S.$$

If X is a mart, equality holds.

$\Rightarrow t \in$ interval

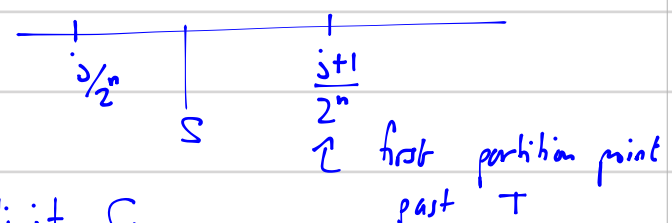
Thm 2.14 (Doob's Optional Stopping Thm (I)) Let $(X_t)_{t\geq 0}$ be R-cts process and $S \leq T$ stopping times.

(1) If (X_t) is a supermart & $S \leq T \leq C$, $C > 0$, then
$$\mathbb{E}[X_T \mid \mathcal{F}_S] \leq X_S$$

(2) If instead (X_t) is a mart, $\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S$.

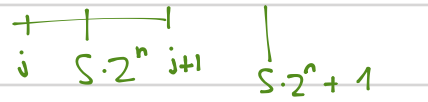
Pf [Case 2 only, (1) is ex.]

Let $S_n(\omega) = \frac{1}{2^n}\left[S(\omega)\cdot 2^n + 1\right]$



Then S_n are S.T.s decreasing with n, with limit S.

$$[S_n \leq t] = \left[\omega : S(\omega) \leq \frac{m}{2^n}\right] \text{ if } t \in \left[\frac{m}{2^n}, \frac{m+1}{2^n}\right]$$
$$\in \mathcal{F}_{\frac{m}{2^n}} \subset \mathcal{F}_t.$$



$j+1 = [S\cdot 2^n + 1]$
$= \lfloor S\cdot 2^n + 1 \rfloor$

Consider $(X_t,\ t \in \left\{\frac{i}{2^n}\right\}_{i=1}^{\infty})$. By Doob's EOST,

$$X_{S_n} = \mathbb{E}[X_c \mid \mathcal{F}_{S_n}]$$

$(X \in L^1, \ldots) \Rightarrow \{X_{S_n}\}$ are UI

$$\Rightarrow \mathbb{E} \lim_{n\to\infty} X_{S_n} \overset{UI}{=} \lim_{n\to\infty} \mathbb{E} X_{S_n} = \mathbb{E} X_c$$

Apply thm 2.12 to conclude $\mathbb{E}[X_c \mid \mathcal{F}_S] = X_S$.

□

Recall

- An L^1-bdd supermart (X_t) has a limit as $t \to T (\text{or } \infty)$, written $X_T \in L^1(X_\infty)$
- (X_t) is a mart $\Rightarrow (X^S_t)$ is a mart (S is a s.t.)

Thm 2.9 If $(X_t : t \in [0,T))$, $T \in \mathbb{R} \cup \{\infty\}$ is a right cts martingale. TFAE:

(1) X_t converges to a r.v. X_T in L^1. ($\lim_{t\to T} |X_t - X_T| = 0$)

(2) $\exists$ an L^1 r.v. X_T s.t. $X_t = \mathbb{E}[X_T \mid \mathcal{F}_t]$

(3) (X_t) is a U.I. family & $X_t \to X_T$ in probability ($\equiv$ a.s. as is mart.)

Pf ($T=\infty$) In all 3 cases, (X_t) is bdd in L^1 and X_∞ exists & is in L^1 by MartCT.

(1)$\Leftrightarrow$(3) is standard: If $X_t \to X_T$ in $\mathbb{P}$, then $X_t \xrightarrow{L^1} X_T$ is equiv to $\{X_t \text{ is UI}\}$

(2)$\Rightarrow$(3) Given $X_T \in L^1$, $\{\mathbb{E}[X_T \mid \mathcal{G}] : \mathcal{G} \subseteq \mathcal{F}\}$ is U.I.

(3)&(1)$\Rightarrow$(2) $\forall\, t < u$, $\mathbb{E}[X_u \mid \mathcal{F}_t] = X_t$. Since $X_u \xrightarrow{L^1} X_T$,

$$X_t = \lim_{u \to T} \mathbb{E}[X_u \mid \mathcal{F}_t] = \mathbb{E}[X_T \mid \mathcal{F}_t] \qquad \square$$

Thm 2.10 [Doob's Optional Stopping Thm for U.I. processes.]

Let (X_t) be a right-cts, UI supermart. Then $\forall$ pairs of s.t.s $S \le T$,

$$X_T \ge \mathbb{E}(X_\infty \mid \mathcal{F}_T)$$
$$X_S \ge \mathbb{E}(X_T \mid \mathcal{F}_S)$$

with equality if X is a mart.

$X_T(\omega) := X_{T(\omega)}(\omega)$; if $T=\infty$, $X_\infty(\omega)$

$\mathcal{F}_T$? $\mathcal{F}_t = \mathcal{F}_t^X = \sigma(X_s : s \le t)$

$\mathcal{F}_T = \sigma(X_T)$

$\nexists$ $\text{im}\, T \subseteq \mathbb{N}$. $\{T = n\}$: $\mathcal{F}_T = \{A : A \cap [T=n] \in \mathcal{F}_n\}$

Pf (Martingales)

We know $X_\infty = \lim X$ exists a.s.

Also, $X_t = \mathbb{E}[X_\infty \mid \mathcal{F}_t]$. We discretize:

Let $S_n := \sum_{k \ge 0} \frac{k}{2^n} \mathbb{1}_{A_n^k}(\omega)$, $A_n^k := \{\omega : S(\omega) \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\}$. $S_n \downarrow S$, $|S_n - S| = \frac{1}{2^n}$

$\mathbb{E} X_S \approx \mathbb{E} X_{S_n} \approx \mathbb{E} X_{S_1} \mathbb{1}_{A_n^k}$

If $A \in \mathcal{F}_{S_n}$, $\mathbb{E} X_{S_n} \mathbb{1}_{A_n^k} \mathbb{1}_A = \mathbb{E} X_{k/2^n} \mathbb{1}_{A_n^k} \mathbb{1}_A = \mathbb{E} X_\infty \mathbb{1}_{A_n^k} \mathbb{1}_A$

$A \cap \{S_n = \frac{k}{2^n}\} \in \mathcal{F}_{k/2^n}$; $\mathbb{1}_{A_n^k} \mathbb{1}_A \le 1$

Summing over k, $\mathbb{E} X_{S_n} \mathbb{1}_A = \mathbb{E} X_\infty \mathbb{1}_A \Rightarrow X_{S_n} = \mathbb{E}[X_\infty \mid \mathcal{F}_{S_n}] \Rightarrow X_{S_n}$ is U.I.

$A \in \mathcal{F}_S \subseteq \bigcap \mathcal{F}_{S_n}$,

$$\mathbb{E} X_S \mathbb{1}_A = \lim \mathbb{E} X_{S_n} \mathbb{1}_A = \mathbb{E} X_\infty \mathbb{1}_A \Rightarrow X_S = \mathbb{E}[X_\infty \mid \mathcal{F}_S] \qquad \square$$

Cor (1) If M_t is a R-cts mart, then $\{M_T : T \text{ bdd s.T.}\}$ is UI.
(2) If $(M_t : t \le \infty)$ is UI R-cts mart, then $\{M_T : T \text{ any s.T.}\}$ is UI.

Pf (1) $M_t = \mathbb{E}[M_\infty \mid \mathcal{F}_T]$ (2) $M_T = \mathbb{E}[M_\infty \mid \mathcal{F}_T]$ □

Rem (M_t) is a mart. IFF $\forall$ bdd s.t. T, $\mathbb{E}M_T = \mathbb{E}M_0$
That $\mathbb{E}M_t$ is const. does not imply M is mart.

(12)

---

eg If (X_t) is a cts process, $T_n = \inf_t\{|X_t| \ge n\}$ is a S.T., $(X_t^{T_n})$ is a UI mart $\forall n$
$T_n \le T_m$ if $n \le m$, $\lim_{n\to\infty} T_n(\omega) = \infty$

§5: Local Martingales

Defn (1) Let (X_t) be an adapted process. $\exists$ a sequence of stopping times T_n st $T_n \le T_m$ a.s. and $\lim_{n\to\infty} T_n = \infty$ and $\forall n$,

$$X^{T_n} - X_0 \qquad \text{for: } X^{T_n}\mathbb{1}_{T_n>0}$$

is a U.I. martingale, we say (X_t) is a local martingale and (T_n) reduces (X_t).

(2) An n-dim^l stoch. process $(X_t^1, \dots, X_t^n)$ is a local mart. if each component is.

(3) A semi-martingale is an adapted stoch. process of the form

$$X_0 + M_t + A_t$$

where X_0 is a r.v., (M_t) a local martingale, & (A_t) a proc of finite total variation, with $M_0 = 0$, $A_0 = 0$ Semi-marts are 'stable for integration'

Remark (1) A local martingale is a martingale if $\forall t$, $\{M^T : T \le t \text{ s.T.}\}$ is UI.
(2) ——— if $|M_t| \le Z$ for some $Z \in L^1$.
(3) If M_t is a local mart with $M_0 = 0$, we do not know $\mathbb{E}M_t = 0$!

Motivation

If (M_t) is a local mart, we will see $f_t \in L^2_{loc}(M_t) \Rightarrow \int_0^t f_s \, dM_s$ is a loc. mart.

If a_t is an increasing process, (B_{a_t}) is expected to be a local mart.

$\int_0^t f_s \, dB_s$ is a local mart.

If (M_t) is a local. cts mart, $\langle M, M \rangle_\infty = \infty$ $B_{\langle M, M \rangle_t} = M_t$.

Thm 2.11 Let (X_t) be a simple cts local mart. Let

$$A = \{ \omega : (X_\bullet(\omega))_{TV}([a,b]) < \infty \}$$

Then $X_t(\omega) = X_a(\omega)$ a.s. for $\forall a \le t \le b$ on A loc. mart don't have $TV < \infty$

$$X_{TV}([a,b]) := \sup_{\Delta} \sum_{1}^{N+1} \left| X_{t_{i+1}} - X_{t_i} \right|^p \text{ for } p^{th} \text{ variation}$$

, sup over partitions $\Delta = \{a = t_0 < t_1 < \cdots < t_N = b\}$

Def 2.4 Let (X_t) & (Y_t) be 2 cts stoch processes. If for any Δ_n, sequence of partitions of $[0,T]$ with $|\Delta_n| \xrightarrow[n \to \infty]{} 0$

$$\lim_{n \to \infty} \sum_{j=0}^{n} \left(X_{t_{j+1}^{(n)} \wedge t} - X_{t_j^{(n)} \wedge t} \right) \left(Y_{t_{j+1}^{(n)} \wedge t} - Y_{t_j^{(n)} \wedge t} \right) \text{ exist in probability}$$

We define the limit to be the bracket of X & Y, $\langle X, Y \rangle_t$.

$$(M_t)^2 = \underbrace{(M_0)^2 + 2 \int_0^t M_s \, dM_s}_{\text{Bdd var enough}} + \langle M, M \rangle_t$$

'integration by parts'

$$M_t N_t = M_0 N_0 + \int_0^t M_s \, dN_s + \int_0^t N_s \, dM_s + \langle M, N \rangle_t$$

$\langle M, M \rangle_t = 0$ if M has bdd var.

missed lecture (15) (Source: Febin)

Thm 2.12 (5) For any cts local martingales (N_t) & (M_t), $\exists!$ cts part of finite variation vanishing at 0, denoted by $\langle M, N\rangle$ s.t.

$$M_t N_t - \langle M, N\rangle_t \text{ is a local mgl.}$$

This process is called the masking bracket of (M_s) & (N_s), or the quadratic variation process.

---

you fix ω. For a.e. ω, $t \longmapsto \langle M, N\rangle_t(\omega)$ everything is sample cts.

(2) $$\langle M, N\rangle_t = \frac{1}{4}\langle M+N,\ M+N\rangle_t - \frac{1}{4}\langle M-N,\ M-N\rangle_t$$

$$\langle M - M_0,\ N - N_0\rangle = \langle M, N\rangle$$

(3) $t \longmapsto \langle M, M\rangle_t$ is an increasing process for a.e. ω.

(4) If (M_t) is a bdd process then $(M_t)^2 - \langle M, M\rangle_t$ is a mgl.

(5) $$\langle M, M\rangle_t \overset{\text{prob}}{=} \lim_{n\to\infty} \sum \left[M_{t_{i+1}^{(n)} \wedge t} - M_{t_i^{(n)} \wedge t}\right]^2$$

You take some partition $\Delta_n = \left(0 = t_0^{(n)} < t_1^{(n)} < \cdots < T\right), \ldots, |\Delta_n| \longrightarrow 0$

(6) $(M, N) \longmapsto \langle M, N\rangle$ is a bilinear process, also $\langle M, N\rangle = \langle N, M\rangle$.

eg Let (B_t) be a BM. Then $\langle B, B\rangle_t = t$.

(Pf) $$\mathbb{E}\left[B_t^2 - B_s^2 \mid \mathcal{F}_s\right] = \mathbb{E}\left[(B_t - B_s)^2 \mid \mathcal{F}_s\right]$$
$$\Longrightarrow \mathbb{E}(B_t - B_s)^2 = t - s$$

Thm (Levy Martingale Characterization Theorem.)

An $(\mathcal{F}_t)$-adapted sample cts process (X_t) with $X_0 = 0$ is a std BM iff it is an $\mathcal{F}_t$-mgl with bracket $\langle X, X\rangle_t = t$.

H/W If X, Y are indep processes then $\langle X, Y\rangle = 0$.

- if X_t is a Finite Variation Process (Finitive?) (Y_t) is a semi-martingale, then $\langle X, Y\rangle_t = 0$.

- $\langle M^1 + A^1, \underbrace{M^2 + A^2}_{\text{semi-mgl}}\rangle_t = \langle M^1, M^2 + A^2\rangle + \langle A^1, M^2 + A^2\rangle$

$= \langle M^1, M^2\rangle_t$ as the others are 0.

Preview If I defined $\int_0^t f_s\, dM_s$, $\int_0^t g_s\, dN_s$ (local mgl)

and I want to compute

$$\left\langle \int_0^t f_s\, dM_s,\ \int_0^t g_s\, dN_s \right\rangle.$$

We can also write this as

$$\left\langle \int_0^\bullet f_s\, dM_s,\ \int_0^\bullet g_s\, dM_s \right\rangle_t = \int_0^t g_s f_s\ d\langle M, N\rangle_s.$$

Riemann-Stieltjes Integral.

we have not defined bracket for M not cts

Thm 2.14 (1) If M_t is a bdd (cts) mgl with

$$\langle M, M\rangle_t = \langle M, M\rangle_a$$

for some $t > a$, then $M_s = M_a$ a.e. $(a \le s \le t)$

(If your bracket does not increase over an interval then the mgl must be const in t.)

(2) Let (M_t), (N_t) be cts local mgl & T a s.t. then $\langle M^T, N^T\rangle_t = \langle M, N\rangle_t^T$
$= \langle M^T, N\rangle_t$

Pf (1) Let $a < s \le t$. Recall $(M_s)^2 = \langle M, M\rangle_s$ is a mgl.

$$\mathbb{E}[M_s - M_a]^2 = \mathbb{E}[M_s]^2 - \mathbb{E}[M_a]^2 \text{ by mgl property}$$

again mgl property $= \mathbb{E}\langle M, M\rangle_s - \mathbb{E}\langle M, M\rangle_a = 0$

as the bracket is const in that small interval.

since $\mathbb{E}(M_s^2) - \mathbb{E}\langle M, M\rangle_s = \mathbb{E}M_a^2 - \mathbb{E}\langle M, M\rangle_a$

(2) H/W ex first consider bdd case & just use defn. You consider $M^T N^T - \langle M, N\rangle^T$.

(L)

2 cts loc mgl $M, N \Rightarrow \exists!$ cts process of FV $\langle M,N\rangle_t$ st $\langle M,N\rangle_0 = 0$ and $M_t N_t - \langle M,N\rangle_t$ is an mgl.

Mgl Ineq.

Prop If (X_n) is a sub-mgl ($\mathbb{E}X_n^- < \infty$ suffices) then $\forall \lambda > 0$, $N \in \mathbb{N}$.

$$\lambda \mathbb{P}\left[\omega : \sup_{1\le n\le N} X_n(\omega) \ge \lambda\right] \le \mathbb{E}\left[X_N \mathbb{1}_{\sup_{1\le n\le N} X_n \ge \lambda}\right]$$

Pf Let $T(\omega) := \begin{cases} \inf\{n : X_n(\omega) \ge \lambda\} & \text{if such } n \text{ exists } \le N \\ N & \text{if } X_n(\omega) \le \lambda \ \forall n \le N. \end{cases}$

T is a s.t. bdd by N.

$$\Rightarrow \mathbb{E}X_N \ge \mathbb{E}X_T = \mathbb{E}\left[X_T \mathbb{1}_{\sup_{1\dots N} X_n \ge \lambda}\right] + \mathbb{E}\left[X_T \mathbb{1}_{\sup_{1\dots N} X_n < \lambda}\right]$$

$$\ge \lambda \mathbb{P}\left[\sup_{1\dots N} X_n \ge \lambda\right] + \mathbb{E}\left[X_N \mathbb{1}_{\sup_{1\dots N} X_n < \lambda}\right]$$

Hence, $\mathbb{E}\left[X_N \mathbb{1}_{\sup_{1\dots N} X_n \ge \lambda}\right] + \mathbb{E}\, X_N \mathbb{1}_{\sup_{1\dots N} X_n < \lambda} \ge \lambda \mathbb{P}\left[\sup_{1\dots N} X_n \ge \lambda\right] + \mathbb{E}\left[X_N \mathbb{1}_{\sup X_n \ge \lambda}\right]$

□

Cor If X_n is a mgl or a +ve sub-mgl then

(X mgl $\Rightarrow |X|$ is sub-mgl.)

(1) for every $p \ge 1$ & $\lambda > 0$, $\lambda^p \mathbb{P}\left[\sup_{1\dots N} |X_n| \ge \lambda\right] \le \mathbb{E}\left[|X_N|^p\right]$

(2) for every $p > 1$,

$$\mathbb{E}\left[\sup_{1\dots N} |X_n|^p\right] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|X_N|^p\right]$$

Controlled by last term

Pf Renwz - Yor Ch 2 Cor 1.6.

when do we use cadlag?

Thm 2.16 Let $(X_t : t \in I)$, I an interval, be a right cts mgl OR right cts $+^{ve}$ sub-mgl. Then

(1) $\forall p \geq 1$, $\lambda > 0$,

$$\lambda^p \mathbb{P}\left[\sup_{t\in I} |X_t| \geq \lambda\right] \leq \sup_{t\in I} \mathbb{E}|X_t|^p \quad \text{Maximal Ineq.}$$

(2) L^p inequality $\forall p \geq 1$,

i.e. Doob's ineq. $\mathbb{E}\left[\sup_{t\in I} |X_t|^p\right] \leq \left(\frac{p}{p-1}\right)^p \sup_{t\in I} \mathbb{E}|X_t|^p$

Pf Note (X_t) is determined by $(X_q, q \in I \cap \mathbb{Q})$. □

Thm 2.17 [Burkholder–Davis–Gundy Inequality]

else subtract M_0

$\forall T \leq \infty, \forall p > 0$, $\exists$ constants $c_{p>0}$ & $C_{p>0}$ st. $\forall$ cts loc. mgls (M_t) with $M_0 = 0$

$$c_p \mathbb{E}\left[\langle M, M\rangle_T^{p/2}\right] \leq \mathbb{E}\left[\sup_{0\leq t<T} |M_t|^p\right] \leq C_p \mathbb{E}\left[\langle M, M\rangle_T^{p/2}\right]$$

$$\left(= C_p \mathbb{E}\left[\langle M, M\rangle_T\right]^{p/2} \text{ notation}\right)$$

Thm 2.18 [Lévy's Mgl Charact^n Thm]

A $(\mathcal{F}_t)$-adapted stoch. process (B_t) with values in $\mathbb{R}^n$ and $B_0 = 0$ is an $(\mathcal{F}_t)$-BM iff $B_t = (B_t^1, \dots, B_t^n)$ is a mgl and $\forall i, j \in [n]$

$$\langle B^i, B^j\rangle = t\,\delta_{i,j}$$

(?) $\langle \cdot, \cdot\rangle = 0 \Rightarrow \mathbb{E}[\cdot \times \cdot] = 0$

Thm [Dubins–Schwartz Thm] Let (M_t) be a cts loc. mgl with $M_0 = 0$ st. $\langle M, M\rangle_\infty = 0$

Define for each $t > 0$, $T_t = \inf\{s : \langle M, M\rangle_s > t\}$, $T_0 = 0$. Then

$$\left(M_{T_t} : t \geq 0\right)$$

is an $(\mathcal{F}_{T_t})$-BM and (time-inverse)

$$M_t = B_{\langle M, M\rangle_t}$$

(L)

Recall $f \in L^1 \Rightarrow$ we can choose a version $Y_t = \mathbb{E}[f \mid \mathcal{F}_t]$ s.t. Y_t is right-cts. ($\sup_t \|M_t\|_2 < \infty$)

Thm Let $\mathcal{H}^2$ be the space of L^2 bdd right-cts martingales. Let

$$\|M\|_{\mathcal{H}^2} = \sqrt{\mathbb{E}(M_\infty^2)} = \lim_{t\to\infty} \sqrt{\mathbb{E} M_t^2}.$$

(a process)

Then $\mathcal{H}^2$ is a Hilbert Space. Furthermore, let $H^2 \leq \mathcal{H}^2$ be the subspace of cts (not just r-cts) L^2-bdd mgls. Then H^2 is a closed subspace of $\mathcal{H}^2$.

Pf (1) If $(M_t) \in \mathcal{H}^2$, $M_\infty = \lim M_t$ exists, and $M_t = \mathbb{E}[M_\infty \mid \mathcal{F}_t]$.

By Doob's inequality,

$$\mathbb{E} \sup_{t<\infty} (M_t)^2 \leq \sup_{t<\infty} \mathbb{E}(M_t^2) < \infty.$$

$\Rightarrow (M_t)^2$ is UI.

$\Rightarrow M_t^2 \xrightarrow{L^1} M_\infty^2$ so $M_\infty \in L^2$

So $\exists$ 1-1- corresp. of $\mathcal{H}^2$ with $L^2(\Omega, \mathcal{F}, P)$:

$$(M_t) \longmapsto M_\infty$$
$$(\mathbb{E}[f \mid \mathcal{F}_t]) \longleftarrow f$$

That preserves the norm

Further, $\mathbb{E} M_\infty^2 = \mathbb{E}\langle M, M\rangle_\infty \overset{\text{Martingale convergence}}{=} \lim \mathbb{E}\langle M, M\rangle_t$ ($M_t^2 - \langle M, M\rangle_t$ is UI)

($(M_t)^2 - \langle M, M\rangle_t$ is UI mgl.)

(2) Let $M_t^n \in \mathcal{H}^2$, $\lim_{n\to\infty} \|M^n - M\|_{\mathcal{H}^2} = 0$ for some $M \in \mathcal{H}^2$. STP M_t cts

$$\mathbb{E} \sup_t (M_t^n - M_t)^2 \leq 4 \underbrace{\sup_t \mathbb{E}|M_t^n - M_t|^2}_{\|M^n - M\|_{\mathcal{H}^2}} \longrightarrow 0$$

so L^2 converges. $\Rightarrow$ converges in P $\Rightarrow$ $\exists$ subseq. $\{n_k\}$ st

$$\sup_t \left| M_t^{n_k}(\omega) - M_t(\omega) \right| \longrightarrow 0 \quad \text{for a.e. } \omega$$

Hence $t \longmapsto M_t(\omega)$ is cts for a.e. ω. $\square$

H/W If (M_t) is a cts local mgl with $M_0 = 0$, TFAE:

(1) $(M_t) \in H^2$ (M_t is L^2 bdd cts)

(2) $\mathbb{E}\langle M, M\rangle_\infty < \infty$.

Further, if either hold then $M_t^2 - \langle M, M\rangle_t$ is a U-I mgl.

Hint Let T_n be a reducing sequence of stopping times.

$\Longrightarrow \mathbb{E}(M_t^{T_n})^2 = \mathbb{E}[\langle M, M\rangle_t^{T_n}]$, justify $n \to \infty$.

Ch 3 STOCHASTIC INTEGRALS

recall elementary processes,

$$\mathcal{E} = \left\{ H \;\middle|\; \forall\, 0 = t_0 < t_1 < \cdots \lim_{n\to\infty} t_n = \infty, \quad H_t(\omega) = H_{-1}(\omega)\mathbb{1}_{\{0\}}(t) + \sum_0^\infty H_i(\omega)\mathbb{1}_{(t_i, t_{i+1})}(t), \quad H_i \in \mathcal{F}_{t_i}, \ \sup_{\omega,i} |H_i(\omega)| < \infty \right\}$$

Elementary integral w.r.t. a stoch. process X_t is

$$\int H_s\, dX_s = H_{-1}(\omega)X_0(\omega) + \sum_1^\infty H_i(\omega)\left[X_{t_{i+1}\wedge t}(\omega) - X_{t_i \wedge t}(\omega)\right]$$

Q1 if $H_n \in \mathcal{E}$ & $H^{(n)} \to f$ in some sense,

$\int_0^t H_s\, dX_s \to I_t(H)$ in some sense.

we define $\int_0^t f\, dX_s = \lim_{n\to\infty} \int_0^t H_s^{(n)}\, dX_s$

Is $H \mapsto \int_0^t H_s\, dX_s$ cts? for which (X_t) is it cts?

'ucp' uniform conv in $\mathbb{P}$

Partial answer: $X_t = A_t + M_t$ Cadlag semi-martingale, Answer is yes.

Q2 $\overline{\mathcal{E}}^{ucp} = ?$ Partial answer $\overline{\mathcal{E}}^{L^2(X)} \supseteq$ Left cts adapted process $=$ progressively meas process.

$= \mathbb{R}$ or $\mathbb{R}^n$ (eg.)

Def A stoch. process on E metric space $(X_t : t \geq 0)$ is progressively meas. if

(1) $X : \mathbb{R}_+ \times \Omega \longrightarrow E$ is meas. (wrt $\text{Borel}(\mathbb{R}^+) \otimes \mathcal{F}_\infty$)

(2) $X : [0,t] \times \Omega \longrightarrow E$ is meas wrt. $\text{Borel}([0,t]) \otimes \mathcal{F}_t$

left cts adpted or right cts adapted are prog. meas.

Bracket process nice for cts process, so we assume ch $(M_t) \in H^2$. Define integral first then 'localize'.

$$L^2(M) = \left\{ f : [0,T] \times \Omega \to \mathbb{R} \text{ prog-meas. process}, \underbrace{\mathbb{E}\left[\int_0^T (f_s^{(\omega)})^2 \, d\overbrace{\langle M, M\rangle_s}^{\text{increasing (Stieljes integral)}}(\omega)\right]}_{\|f\|^2_{L^2(M)}} < \infty \right\}$$

$L^2(M)$ is a Hilbert space, with Hilbert space norm $\|f\|_{L^2(M)}$.

— is actually a std L^2 space: if Γ is a meas. subset of $[0,T]\times\Omega$, $\mu_{\langle M,M\rangle}(\Gamma) = \mathbb{E}\left[\int_0^T 1_\Gamma \, d\langle M,M\rangle_s\right]$.

– think of $[0,T]$ as a cpct interval ie. $T<\infty$, or $[0,\infty)$. $T=\infty$ case covers other cases.

Thm 3.1 $\mathcal{E}$ is dense in $L^2(M)$, for $M \in H^2$.

Proof Let $H_t = 1_{\{0\}}(t) H_{-1} + \sum_i^\infty H_i 1_{(t_i, t_{i+1}]}(t)$, $\sup_i H_i$ bdd ($\sum_i^\infty$ is a finite sum)

$H_i \in \mathcal{F}_{t_i}$

$H_{-1} \in \mathcal{F}_0$.

$$\|H\|^2_{L^2(M)} = \sum_i^\infty \mathbb{E}\, \underbrace{H_i(\omega)^2}_{\text{unif bdd}} \left(\langle M\rangle_{t_{i+1}\wedge t} - \langle M\rangle_{t_i \wedge t}\right) \quad \langle M\rangle := \langle M, M\rangle$$

$$\leq C \langle M\rangle_\infty < \infty$$

We only prove $\mathcal{E}$ is dense in $L^2(M) \cap \{\text{left cts processes}\}$

↳ I will tell you later how to fix.

- First assume f is bdd & $\mathcal{F}$-cts.

  Let $f_n(t,\omega) = f_0(\omega) 1_{t=0} + \sum_i^\infty \underbrace{f_{t_i}(\omega)}_{\text{left value}} 1_{(t_i, t_{i+1}]}(\omega)$

  where $t_i \in \left\{\frac{k}{2^n}, k = 0,1,2,\dots\right\}$. Then $f_n(t,\omega) \xrightarrow[n\to\infty]{} f(t,\omega)$ a.s. ω.

  Since $|f_n|_\infty \leq |f|_\infty$, by the DCT $f_n \xrightarrow{L^2} f$.

- If f is not bdd, Let $(f_N)_s(\omega) = f(\omega) 1_{\{|f_s(\omega)\leq N\}}$

  Then $|f_N - f|^2_{L^2(M)} \leq \mathbb{E}\int_0^\infty f(\omega) 1_{|f|>N} \, d\langle M\rangle_s^{(\omega)} \xrightarrow[N\to\infty]{} 0$ □

Remark There are nice approx. of left cts process by 'simple predictable process' replace t_i with S.T.s

Thm 3.3 Let $(M_t) \in H_0^2$ (start at 0, cts, L^2-bdd)

Then for $H \in \mathcal{E} \leq L^2(M)$,

$$H \longmapsto \int_0^\bullet H_s \, dM_s \in H_0^2$$

Further, this map is an isometry.

EXERCISE similar to 3.1 proof.

Thm + Defn 3.4 If $f \in L^2(M)$ and f_n is a sequence in $\mathcal{E}$ converging to f in $L^2(M)$, then $\int_0^\bullet f_n(s) \, dM_s$ converges in H_0^2. We define the limit to be

$$\int_0^\bullet f_s \, dM_s$$

(the limit is indep. of the sequence $\{f_n\}$)

(L) (2 hrs missed on wednesday (Thm matt))

$M_t \in H_0^2$; $M_t^2 - \langle M, M\rangle_t$ is a mgle

$\mathbb{E}\langle M, M\rangle_t = \mathbb{E}(M_t)^2$

$\int_0^t f_s \, dM_s \quad f \in L^2(M), \quad \mathbb{E}\int_0^\infty [f_s(\omega)]^2 \, d\langle M, M\rangle_s(\omega) < \infty$

$\langle M\rangle_s := \langle M, M\rangle_s \quad (\langle M\rangle_s(\omega), \ r \geq 0) \longrightarrow \mu$ a measure on $\mathbb{R}_+$.

$\langle M, N\rangle_s$, M, N (cts loc. mgl) each ω.

$\langle M, N\rangle_s(\omega) \longrightarrow$ a signed mes $\mu_{\langle M,N\rangle}(\omega)$ on $\mathbb{R}_+$

$\langle M, N\rangle_t = \langle M, N\rangle_t^+ - \langle M, N\rangle_t^- \longleftrightarrow \mu = \mu^+ - \mu^- \qquad |\mu| = \mu^+ + \mu^-$

$|\langle M, N\rangle_t^s| \leq \sqrt{\langle M\rangle_t^s \langle N\rangle_t^s}$ $\left(\langle M, M\rangle_t^s := \langle M, N\rangle_t - \langle M, N\rangle_s = : \mu_{\langle M,N\rangle}([s, t])\right)$.

Prop Let (M_t) & (N_t) be cts loc. mgls.
(H_t) & (K_t) be measurable processes by which we mean

$H, K : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$ are Borel measure (measurable?)

Then (1) $\int_0^t |H_s| |K_s| \, |d\langle M, N\rangle_s| \leq \sqrt{\int_0^t H_s^2 \, d\langle M, M\rangle_s} \sqrt{\int_0^t K_s^2 \, d\langle N, N\rangle_s}$ ("Cauchy-Schwarz")

(2) $\left|\int_0^t H_s K_s \, d\langle M, N\rangle_s\right| \leq \sqrt{\int_0^t H_s^2 \, d\langle M, M\rangle_s} \sqrt{\int_0^t K_s^2 \, d\langle N, N_s\rangle}$

Indication of proof Let (H), (K) be elementary processes.

Proof Since $f_n \xrightarrow{L^2(M)} f$,

$$\left|\int_0^\cdot f_n(s) \, dM_s - \int_0^\cdot f_m(s) \, dM_s\right|_{H^2}$$
$$= \left|\int_0^\cdot (f_n(s) - f_m(s)) \, dM_s\right|_{H^2}$$
$$= \left|\mathbb{E}\int_0^\infty [f_n(s) - f_m(s)]^2 \, d\langle M, M\rangle_s\right|$$
$$= |f_n - f_m|_{L^2(M)} \xrightarrow{n,m\to\infty} 0 \text{ since Cauchy} \Rightarrow \int_0^\cdot f_n(s) \, dM_s \text{ Cauchy}$$

and it has a limit.

Uniqueness is standard. let $g_n \xrightarrow{L^2(M)} f$.

$$\left| \int_0^\cdot g_n \, dM_s - \int_0^\cdot f_n \, dM_s \right|_{H^2} \le \int_0^\infty (g_n - f)^2 \, d\langle M \rangle_s + \int_0^\infty (f_n - f)^2 \, d\langle M \rangle_s$$

$$\xrightarrow[n \to \infty]{} 0$$

So the limit is indep of choice of sequence $\mathcal{E} \to f$.

Thm Let $M \in H^2$, $K \in L^2(M)$; $\exists!$ etc $I(K,M) \in H_0^2$ se $\forall N \in H^2$,

$$\langle I(K,M), N \rangle_t = \int_0^t K_s \, d\langle M, N \rangle_s \quad \forall t \ge 0$$

equivalently, $\langle I(K,M), N \rangle_\infty = \int_0^\infty K_s \, d\langle M, N \rangle_s$ ——— (★)

Furthermore, $K \in L^2(M) \longrightarrow I(K,M) \in H_0^2$ is an L^2-isometry & $I(K,M)$ agrees with the Stochastic integral

st $\mathbb{E}(I,N)_\infty \overset{(\#)}{=} U(N) = \mathbb{E} \int_0^\infty K_s \, d\langle M, N \rangle_s$

From this we prove (★): Let $X_t := I_t N_t - \int_0^t K_s \, d\langle M, N \rangle_s$

We prove below (X_t) is a mgl & by the defn of the bracket proc.

$$\langle I, N \rangle_t = \int_0^t K_s \, d\langle M, N \rangle_s .$$

Let τ be a bdd stopping time.

$$\mathbb{E}[I_\tau N_\tau] : \mathbb{E}\langle I, N \rangle_{\tau \wedge \infty} = \mathbb{E}\langle I, N^\tau \rangle_\infty$$

since $IN - \langle I, N \rangle$ is an mgl.

by (#), $\mathbb{E} \int_0^\infty K_s \, d\langle M, N^\tau \rangle_s = \mathbb{E} \int_0^\tau K_s \, d\langle M, N \rangle_s$

$\Longrightarrow \mathbb{E} X_\tau = 0$ so (X_t) is a mgl $\Longrightarrow$ (★).

It is clear (ex) that $K \mapsto I(K, M)$ is linear;

Let $K, K' \in L^2(M)$: $\langle I(K, M), I(K', M) \rangle_{H_0^2}$

$= \mathbb{E}\big(I(K, M), \underbrace{I(K', M)}_{N_t H_0^2}\big)_\infty$

$= \mathbb{E} \int_0^\infty K_s \, d\langle M, I(K', M) \rangle_s$

$\langle M, I(K', M) \rangle$

$= \langle I(K', M), M \rangle = \int_0^\cdot K'_s \, d\langle M \rangle_s.$

$\mathbb{E} \int_0^\infty K_s K'_s \, d\langle M, M \rangle_s = \langle K, K' \rangle_{L^2(M)} \Rightarrow K \mapsto I(K, M)$ is an isometry $\square$

(L) (Febin)

Prop For $M, N \in H^2_0$, $H \in L^2(M)$, $K \in L^2(N)$. Then

$$\left\langle \int_0^\bullet H_s \, dM_s, \int_0^\bullet K_s \, dN_s \right\rangle = \int_0^\bullet H_s K_s \, d\langle M, N\rangle_s$$

(ex) We have proved a similar result last last time, prop 3.7 localization:

Let T be a s.t., $M \in H^2_0$, $K \in L^2(M)$. Then

$$\int_0^{t\wedge T} K_s \, dM_s = \int_0^t 1_{s<T} K_s \, dM_s = \int_0^t K_s \, dM_s \quad \text{(H/W)}$$

Cor If S, T are s.t.s, $S \le T$. Then on $[t \le s]$,

$\int_0^t K_s \, dM^S_s = \int_0^t K_s \, dM^T_s$; this follows from last pp.

Def Let (M_t) be a loc. mgl cts (cadlag: decompose into cts & jump pt)

$$L^2_{loc}(M) = \{ (H_t) \text{ progressively measurable} : \int_0^t (H_s)^2 \, d\langle M, M\rangle_s < \infty \text{ a.e. } \forall t > 0 \}$$

Any cts process $L^2_{loc}(M)$

Thm Let (M_t) be a cts loc. mgl, $H \in L^2_{loc}(M)$; then $\exists!$ cts loc. mgl $I(H, M)$ with $I(H,M)(0) = 0$ a.e. st $\forall$ cts mgl (N_t), the following holds for any $t \ge 0$:

$$\langle I(H,M), N\rangle_t = \int_0^t H_s \, d\langle M, N\rangle_s \quad \text{(a.e.)} \qquad (\star)$$

Pf (1) Uniqueness If I_1, I_2 are cts loc. mgls with $I_1(0) = I_2(0) = 0$ satisfying $(\star)$.

Let T_n be a sequence of s.t.s st $I_1^{T_n}$ & $I_2^{T_n}$ are bdd mgls: Then $\forall N \in H^2_0$,

know $\langle I_i^{T_n}, N\rangle_t = \langle I_i, N\rangle_t^{T_n} = \int_0^{t\wedge T_n} H_s \, d\langle M, N\rangle_s$, $i = 1, 2$.

$$\Rightarrow \langle I_1^{T_n}, N\rangle_t = \langle I_2^{T_n}, N\rangle \quad \forall N \in H^2$$

Since $I_1^{T_n}, I_2^{T_n} \in H^2$ we have $I_1^{T_n} = I_2^{T_n}$ (a.s.) Since $T_n \uparrow \infty$,

$$I_1 = I_2 \quad \text{(a.s.)}.$$

(2) Existence. Let $T_n = \inf\{t \ge 0 : \int_0^t (1 + H_s^2) \, d\langle M, M\rangle_s \ge n\}$.

T_n is seq. of increasing s.t.s. Observe

$$\int_0^{t\wedge T_n} (H_s)^2 \, d\langle M^{T_n}, M^{T_n}\rangle_s \le n \quad \text{so } M^{T_n} \in H^2_0, \; 1_{s \le T_n} H \in L^2(M^{T_n}).$$

$\int_0^t 1_{s \le T_n} H_s \, dM^{T_n}_s$ makes sense: if $n \le m$ on $\{t \le T_n\}$,

$$\int_0^t \mathbb{1}_{s\le T_n} H_s\, dM_s^{T_n} = \int_0^t \mathbb{1}_{s\le T_m} H_s\, dM_s^{T_m} \longrightarrow \text{ on } \{t \le T_n\}$$

(the $n \ge m$ on $\{t \le T_n\}$)

I am defining these on a larger set & they agree. Since $T_n \uparrow \infty$, $\{t \le T_n\} \uparrow \Omega$, $P(\Omega) = 1$ so we are done: it remains to show this object satisfies the conditions of the thm.

Note $I(H,M)_t^{T_n} = \int_0^{t\wedge T_n} H_s\, dM_s = \int_0^t \mathbb{1}_{s\le T_n} H_s\, dM_s^{T_n}$ is a cts mgl $\forall n$

$\Rightarrow$ T_n reduces $I(H,M)$ & $I(H,M)$ is a loc. cts mgl. Let (N_t) be a cts loc. mgl. Let $\{S_n\}$ be a seq. of ST. st. N^{S_n} is a bdd mgl;

$$\langle I(H,M),\, N\rangle_t^{S_n\wedge T_n} = \langle I(H,M)^{T_n},\, N^{S_n}\rangle_t$$

char Thm $\Longrightarrow$ $\int_0^{t\wedge T_n} H_s\, d\langle M, N\rangle_s^{S_n} = \int_0^{t\wedge T_n\wedge S_n} H_s\, d\langle M,N\rangle_s$

Now take $n\to\infty$ to conclude $\star$, i.e. $\left[\, (I(I,M),N)_t = \int_0^t H_s\, d\langle M,N\rangle_s \quad a.e. \,\right]$

Let $X_t = M_t + A_t$, M_t is cts loc. mgl, A_t cts, finite variation.
Let $f \in L^2_{loc}(M) \cap L^2(dA)$ ≥ cts adapted processes

Define $\int_0^t f_s\, dX_s = \int_0^t f_s\, dM_s + \int_0^t f_s\, dA_s$

Observe This integral preserves type

Prop 3.8 If (K_s) is L-cts, M is a cts mgl (loc) and

$$\int_0^t K_s\, dM_s \overset{\text{Prob}}{=} \lim_{|\Delta_n|\to 0} \sum K_{t_i^n}\left(M_{t_{i+1}^n\wedge t} - M_{t_i^n\wedge t}\right)$$

(only conv. in prob, cannot def as Riemann integral. ⟶ 21 ⟶

week 8 — L22

Prop 3.8 Let $a, b \in \mathbb{R}$, (X_s), (Y_s) cts semi mgl, (K_s^1), (K_s^2) are suitable integrands (e.g. loc. bdd prog. meas. or left cts adapted)

Then

(1) $\int_0^t a K_s^1 + b K_s^2 \, dX_s =$

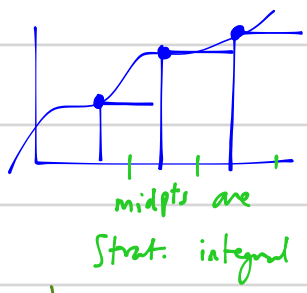
(2) $\int_0^t K_s^1 \, d(aX_s + bY_s) = a \int_0^t K_s^1 \, dX_s + b \int_0^t K_s^1 \, dY_s$

(3) $\int_0^t K_s^1 \left(d \int_0^s K_r^2 \, dX_r \right) = \int_0^t K_s^1 K_s^2 \, dX_s$

We define $\int_s^t = \int_0^t - \int_0^s$

Stratonovich Integral: $\int_s^t K_r \circ dX_r := \int_s^t K_r \, dX_r + \frac{1}{2} \langle K, X \rangle_t$

If (K_s) is left cts $\int_0^t K_r \, dX_r \overset{(P)}{=} \lim_{|\Delta| \to 0} \sum_i K_{\underline{t_i^n}} \left(X_{t_{i+1}^n} - X_{t_i^n} \right)$



Itô's Formula

Thm Let $t \geq 0$, $X_t = (X_t^1, \dots, X_t^d)$ be a cts semi-mgl (each component a semi mgl)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be C^2.

Then for $s \leq t$,

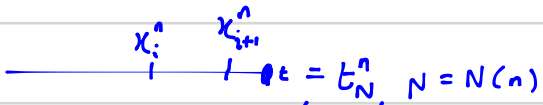
$$f(X_t) = f(X_s) + \sum \int_s^t \frac{\partial f}{\partial x_i}(X_r) \, dX_r^i + \frac{1}{2} \sum_{i,j=1}^d \int_s^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_r) \, d\langle x^i, x^j \rangle_r$$

shorthand: $f(X_t) = f(X_s) + \int_s^t (\nabla f) X_r \, dX_r + \frac{1}{2} \int_s^t \nabla d f(X_r) \, d\langle X, X \rangle_r$ $\quad \langle X, X \rangle = \left(\langle x^i, x^j \rangle \right)_{i,j=1}^d$

Idea of Pf $d=1$, $s=0$. WTS $f(X_t) = f(X_s) + \int_s^t f'(X_r) \, dX_r + \frac{1}{2} \int_s^t f''(X_r) \, d\langle X, X \rangle_r$

Let $x, y \in \mathbb{R}$. $f \in C^2$ so $f(y) = f(x) + f'(x)(y-x) + \int_x^y f''(r)(r-x) \, dr$

$= f(x) + f(x)(y-x) + \frac{f''(x)(y-x)^2}{2} + \int_x^y \left(f''(r) - f''(x) \right)_{(r-x)} \, dr$



$$f(x_t) - f(x_0) = \sum_0^{N(n)-1} \left[f(X_{t_{i+1}^n}) - f(X_{t_i^n}) \right]$$

$$= \underbrace{\sum_0^{N-1} f'(X_{t_i^n}) \left(X_{t_{i+1}^n} - X_{t_i^n} \right)}_{(P) \to \int_0^t f'(X_s) \, dX_s} + \frac{1}{2} \underbrace{\sum f''(X_{t_i^n}) \left(X_{t_{i+1}^n} - X_{t_i^n} \right)^2}_{\text{quad variation}} + \int_{X_{t_i^n}}^{X_{t_{i+1}^n}} \left[f''(r) - f''(X_{t_i^n}) \right] (r - X_{t_i^n}) \, dr$$

$(P) \to \int_0^t f''(X_r) \, d\langle X, X \rangle_r$

assume $|X_t| \leq C$. f'' is unif cts on $[-C, C]$

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ st if $|\Delta_n| < \delta$

$\leq \varepsilon \cdot \text{QuadVar}(X)$; $\varepsilon \downarrow 0$.

(L)

find one
n DAY GOD
GOT SCAMMED

Product Rule

(X_t), (Y_t) semi-martingales $\qquad f(x,y) = xy$

$$X_t Y_t = X_0 Y_0 + \int_0^t \underbrace{Y_s}_{\partial_x f = y}\, dX_s + \int_0^t \underbrace{X_s}_{\partial_y f = x}\, dY_s + \frac{1}{2}\int_0^t 1\, d\langle X,Y\rangle_s + \frac{1}{2}\int_0^t 1\, d\langle Y,X\rangle_s$$
$$\partial_{xy} f = \partial_{yx} f = 1$$

$$X_t Y_t = X_0 Y_0 + \int_0^t Y_s\, dX_s + \langle X, Y\rangle_t \,.$$

Can be def for $\langle X, Y\rangle$
Can prove $\exists\ \langle X, Y\rangle$.

$B_t = (B_t^1 \dots B_t^d)$ — d-BM. $\quad \langle B^i, B^j\rangle_t = t\,\delta_{ij}$

$$f(B_t) = f(B_0) + \int_0^t \sum \frac{\partial f}{\partial x_i}(B_s)\, dB_s^i + \sum_{i=1}^d \int \partial_{x_i x_i} f(B_s)\, ds \,.$$
$$= f(B_0) + \int_0^t (df)_{B_s}\, dB_s + \frac{1}{2}\int_0^t \Delta f(B_s)\, dB_s \,.$$

$f(B_t) - f(B_0) - \frac{1}{2}\int_0^t \Delta f(B_s)\, ds$ is a (local) martingale

$\mathcal{L} = \sum a_{ij}\, \partial_{x_i x_j} + \sum b_i\, \partial_{x_i}$ if a invertible this is Laplacian on some manifold

$f(X_t) - f(X_0) - \int_0^t \mathcal{L} f(X_s)\, ds$.

↑ identifies X_s completely; defines BM on manifold

$f(x) = \frac{1}{x+1}$: Ex work for stopping time

$\dot f(t) = f(t) \Rightarrow f(t) = f(0)e^t$

$dX_t = X_t\, dM_t \qquad X_t = x_0 + \int_0^t X_s\, dM_s$

$X_t = e^{M_t}$? ITÔ $\longrightarrow e^{M_0} + \int_0^t e^{M_s} dM_s + \frac{1}{2}\int_0^t e^{M_s}\, d\langle M,M\rangle_s$

HW: solves Stratonovich.

eg. Let (M_t) be a semi mgl with $M_0 = 1$; $X_t = e^{M_t - \frac{1}{2}\langle M,M\rangle_t}$, (the expon. loc. mgl.)

Then $X_t = X_0 + \int_0^t X_s \, dM_s$

Proof Apply Itô to $Y_t = M_t - \frac{1}{2}\langle M,M\rangle_t$, $f = e^x$

$$f(Y_t) = f\left(M_t - \tfrac{1}{2}\langle M,M\rangle_t\right) \equiv X_t = e^0 + \int_0^t e^{Y_s}\, dY_s + \frac{1}{2}\int_0^t e^{Y_s}\, d\langle Y,Y\rangle_s$$

($e^x = \frac{\partial f}{\partial x}(M_s - \frac{1}{2}\langle M,M\rangle_s)$; $\langle Y,Y\rangle = \langle M,M\rangle$)

$$= 1 + \int_0^t X_s \left(dM_s - \tfrac{1}{2} d\langle M,M\rangle_s\right) + \tfrac{1}{2}\int_0^t X_s \, d\langle M,M\rangle_s$$

$$= 1 + \int_0^t X_s \, dM_s .$$

Ch 4: Stochastic Diff. Equation of Markovian Time

Let $\sigma_i, b : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \dots, m$ Borel measurable. ($\dot{X}_t = b(X_t)$)

or $\sigma_i, b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ ——〃—— ($\dot{X}_t = b(t, X_t)$)

Let $B_t = (B_t^1 \cdots B_t^m)$ be an $\mathbb{R}^n$-valued BM.

equation, $E(\sigma, b)$: $dX_t = \sum_{i=1}^m \sigma_i(t, X_t)\, dB_t^i + b(t, X_t)\, dt$

("$\dot{B}_t\, dt$" white noise)

time indep: $dX_t = \sum_1^m \sigma_i(X_t)\, dB_t + b(X_t)\, dt$

(σ_i: diffusion coeff.; b: drift)

solution: $$X_t = X_0 + \sum \int_0^t \sigma_i(s, X_s)\, dB_s + \int_0^t b(s, X_s)\, ds \quad (\star)$$

ex: (1) Which eqn does $[B_t - z]^{-1}$ satisfy? If it sati. one.

(2) (X_t) satisfies compute $f(X_t)$ and give its semi mgl decomp.

L

Thm 1 Let $(\sigma_i, i \in [m])$, b be Lipschitz cts fns on $\mathbb{R}^d$. Let (B_t) be an $(\mathcal{F}_t)$-BM on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Then $\forall x_0$, $\exists$ a unique $(\mathcal{F}_t)$-adapted ($\mathcal{F}_t \supset \mathcal{F}_t^B$) cts Asch process $(X_t : t \geq 0)$ s.t.

$$X_t = x_0 + \sum_1^m \int_1^t \sigma_i(X_s)\, dB_s^i + \int_0^t b(X_s)\, ds \qquad (\star)$$

By unique we mean if X_t, Y_t satisfies $(\star)$ with $X_0 = Y_0$ a.s., then (X_t) and (Y_t) are indistinguishable (a.s. equal).

Proof Note X_0 is $\mathcal{F}_0$-meas.

$$\int \sigma_i(X_s)\, dB_s^i = \begin{pmatrix} \int \sigma_i^1(X_s)\, dB_s^i \\ \vdots \\ \int \sigma_i^d(X_s)\, dB_s^i \end{pmatrix}$$

[EXISTENCE] Let $X_t^{(0)} = X_0 \quad (\forall t)$

$$X_t^{(1)} = X_0 + \sum_{i=1}^m \int_0^t \sigma_i(X_s^{(0)})\, dB_s^i + \int_0^t b(X_s^{(0)})\, ds$$

$$= X_0 + \sum_1^m \sigma_i(X_0) B_t^i + b(X_0)t$$

$$\vdots$$

$$X_t^{(n+1)} = X_0^{(n)} + \sum_1^m \int_0^t \sigma_i(X_s^{(n)})\, dB_s^i + \int_0^t b(X_s^{(n)})\, ds$$

$$\mathbb{E}|X_t^{(1)} - X_t^{(0)}|^2 \leq C_m \left[\sum_i^m \underbrace{\mathbb{E}|\sigma_i(X_0) B_t^i|^2}_{\mathbb{E}|\sigma_i(X_0)|^2 \, \mathbb{E}|B_t^i|^2} + \mathbb{E}|b(X_0)|^2 t^2 \right]$$

$$|b(x) - b(y)| \leq |b|_{Lip} |x-y| \quad \forall x, y \in \mathbb{R}^d \Rightarrow |b(x)| \leq |b(0)| + |b|_{Lip} |x|$$

$\sigma_i \ldots$

$$\Rightarrow \mathbb{E}|\text{——}|^2 \leq C\, \mathbb{E} \sum_i^m \left[\sigma_i(0) + |\sigma_i|_{Lip} |X_0| \right]^2 + C\, \mathbb{E}\left[|b(0)| + |b|_{lip} |X_0| \right]^2$$

Assume $\mathbb{E}|X_0|^2 < \infty$ (True if $X_0(\omega) = x_0 \in \mathbb{R}^d$)

$$\mathbb{E} \sup_{0 \leq t \leq T} |X_t^{(n+1)} - X_t^{(n)}|^2$$

$$\leq \mathbb{E} \sup_{[0,T]} 2 \left[\underbrace{\sum_1^m \int_0^t \left(\sigma_i(X_s^{(n)}) - \sigma_i(X_s^{(n-1)}) \right) dB_s^i}_{\text{local mart. } M_t^i} \right]^2 + 2 \left[\int_0^t \left(b(X_s^{(n)}) - b(X_s^{(n-1)}) \right) ds \right]^2$$

$\surd$ C-S or $E(X^2) \geq (EX)^2$

B-D-G

$$\leq C \sum \mathbb{E}\langle M^i, M^i\rangle_t + 2T \int_0^T |b(X_s^{(n)}) - b(X_s^{(n-1)}|^2 \, ds$$

$$|\langle M^i, M^i\rangle_T| = \left|\left\langle \int_0^\bullet \sigma_i(X_s^{(n)}) - \sigma_i(X_s^{(n-1)}) \, d\beta_s^i \right\rangle_T\right| = \left|\int_0^T |\sigma_i(X_s^{(n)}) - \sigma_i(X_s^{(n-1)})|^2 \, ds\right|$$

< some discussion about $d > 1$ being hard, $\langle M^i, M^i\rangle_T$ = matrix? >

$$\leq \int_0^T |\sigma_i|_{Lip}^2 \, |X_s^{(n)} - X_s^{(n-1)}|^2 \, ds$$

$$\Longrightarrow \mathbb{E} \sup_{0\leq t\leq T} |X_t^{(n+1)} - X_t^{(n)}| \leq C \int_0^T \mathbb{E} \, |X_s^{(n)} - X_s^{(n-1)}|^2 \, ds \leq C \int_0^T \mathbb{E} \sup_{0\leq r\leq s} |X_r^{(n)} - X_r^{(n-1)}|^2 \, ds$$

where C depends on n, $|\sigma_i|_{lip}$, $|b|_{lip}$ (and possibly d)

$$\leq C^n \int_0^T \cdots \int_0^{t_{n-1}} \mathbb{E} \, |X^{(1)}(t_n) - X^{(0)}(t_n)| \leq \frac{C^n T^{n+1}}{(n+1)!}$$

Lem 1 $T>0$, $t \le T$, $f_s = (f_s^i)_i$ prog meas loc. bdd (adapted L-ctn) stoch process in $\mathbb{R}^d$. $B_t \sim BM(\mathbb{R})$

(1) $\mathbb{E} \sup_{t \le T} \left| \int_0^t f_s \, dB_s \right| \le c \int_0^T \mathbb{E}|f_s|^2 \, ds$

(2) $\qquad ds \Big| \le T \int_0^T \mathbb{E}|f_s|^2 \, ds$

Pf 1 $\mathbb{E} \sup_{t \le T} \left| \int_0^t f_s \, dB_s \right|^2 = \mathbb{E} \sup_{t \le T} \sum_1^d \int_0^t f_s^i \, dB_s^i$

$\le \sum_1^d \mathbb{E} \sup_{t \le T} \left(\int_0^t f_s^i \, dB_s^i \right)^2 \underset{BDG}{\le} c \sum_1^d \mathbb{E} \int_0^T (f_s^i)^2 \, ds = C \int_0^T |f_s|^2 \, ds$

2 C-S works. OR:

$\mathbb{E} \sup_{t \le T} \left| \int_0^t f_s \, ds \right|^2 \le \mathbb{E} \left(\sup_{t \le T} \int_0^t |f_s| \, ds \right)^2 \le T^2 \cdot \mathbb{E} \left(\underbrace{\frac{1}{T} \int_0^T}_{\text{prob measure}} |f_s| \, ds \right)^2$

$\underset{\text{Jensen}}{\le} T^2 \mathbb{E} \frac{1}{T} \int_0^T |f_s|^2 \, ds$

$= T \, \mathbb{E} \int_0^T |f_s|^2 \, ds.$ □

Thm 1 Pf Let $X_t^{(0)} = x_0$

$X_t^{(n+1)} = x_0 + \sum_1^m \int_0^t \sigma_k(X_s^{(n)}) \, dB_s^k + \int_0^t b(X_s^{(n)}) \, ds$ — ✡

(σ_k adapted to $\mathcal{F}^B$ so integral W)

We have from last lecture,

$\mathbb{E} \sup_{s \le t} |X_s^{(n+1)} - X_s^{(n)}|^2 \le C D^n \frac{T^n}{n!}$ ← Fast decay $\Rightarrow$ a.s. (see below)

(C, D constants)

We will prove the sequence $\{ X_\cdot^{(n)} \mid X_\cdot^{(n)}$ is Cauchy in $C([0,T]; \mathbb{R}^d)$ $\mathbb{P}$-a.s. $\}$

$\sup_{s \le T} |X_s^{(n+1)} - X_s^{(n)}|$ is our norm.

$\mathbb{E} \sum_1^\infty |X_s^{(n+1)} - X_s^{(n)}|^2 \overset{\text{Fatou}}{\le} \sum_1^\infty \mathbb{E} \sup |X_s^{(n+1)} - X_s^{(n)}| \le C \sum \frac{(DT)^n}{n!} < \infty.$

$\Rightarrow \exists \, \Omega_0, \ \mathbb{P}(\Omega_0) = 1$ if $\omega \in \Omega_0$, $\sum_1^\infty \sup_{s \le T} |X_s^{(n+1)}(\omega) - X_s^{(n)}(\omega)| < \infty$

$\Rightarrow \{X_s^{(n)}(\omega)\}_n$ is a Cauchy seq.

$\Rightarrow \exists$ a stoch. process $(X_t(\omega), t \le T)$.

st. $X_s^{(n)}(\omega) \xrightarrow[n\to\infty]{} X_s(\omega)$ uniformly in $s \in [0,T]$ as $n \to \infty$

$\Rightarrow (X_s(\omega))$ is cts.

$\Rightarrow X_s$ is meas wrt $\mathcal{F}_s^B$.

We show $(X_s)_s$ is a solution: Take $n \to \infty$ in recurrence (*),

Note $\int_0^T \sigma_k(X_s^{(n)})\, dB_s^k \xrightarrow[n\to\infty]{} \int_0^t \sigma_k(X_s)\, dB_s^k$ in prob

$\Rightarrow \exists$ a subseq. $X_s^{(n_k)}$ which converges a.s. Hence X_t is a soln.

Let (X_t), (Y_t) be 2 solutions with same initial value, $X_0 = y_0$ a.s.

Then (X_t) & (Y_t) are indistinguishable

$$\mathbb{E} \sup_{t\le T} |X_t - Y_t|^2 = \mathbb{E} \sup_{t\le T} \Big| \sum_1^m \int_0^t (\sigma_k(X_s) - \sigma_k(Y_s))\, dB_s^k + \int_0^t |b(X_s) - b(Y_s)|\, ds \Big|^2$$

$$\le 2^m \sum_1^m \mathbb{E} \sup_{t\le T} \Big| \int_0^T (\sigma_k(X_s) - \sigma_k(Y_s)) \Big|^2 dB_s^k + 2^m \mathbb{E} \sup_{t\le T} \Big| \int_0^t (b(X_s) - b(Y_s))\, ds \Big|^2$$

$$\le c 2^m \sum_1^m \mathbb{E} \int_0^T \underbrace{|\sigma_k(X_s) - \sigma_k(Y_s)|^2}_{\le K^2 |X_s - Y_s|^2 \ (\text{Lip})}\, ds \qquad (\text{lem } 1)$$

$$+ 2^m T \int_0^T \mathbb{E} |b(X_s) - b(Y_s)|^2\, ds$$

$$\le c \big[2^m \cdot m + 2^m T\big] K^2 \int_0^T \mathbb{E} |X_s - Y_s|^2\, ds$$

$\int_0^T \mathbb{E} \sup |\text{——}|^2 ds \quad \le c [\text{——}] K^2 \int_0^T \mathbb{E} \sup_{r\le s} |X_r - Y_r|^2\, ds$

By Gronwall's lemma, $\mathbb{E} \sup_{t\le T} |X_t - Y_t|^2 = 0$

$\Rightarrow \sup_{t\le T} |X_t - Y_t| = 0$ a.s. $\square$

$f(T) \le C \int_0^T f(s)\, ds$, $f \ge 0$, f – Borel meas., $\forall t \le T$

Gronwall: $f(t) \le C \int_0^t f(s)\, ds + K \Rightarrow f(t) \le K e^{Ct}$ $\forall t \le T$.

cheated: $\sum_n \sup_{t\le T} |x_t^{(n+1)} - x_t^{(n)}| < \infty$

Minkowski: $\left[\mathbb{E}\left(\sum_n \sup_{t\le T} |X_t^{(n+1)} - X_t^{(n)}|\right)^2\right] \le \sum_1^\infty \left[\mathbb{E} \sup_{t\le T} |X_t^{(n+1)} - X_t^{(n)}|^2\right] < \infty$

Thm 2 As in Thm, but we assume σ_n, b are locally lipschitz cts.
Then $\exists$ a stoch process $(X_t(\omega),\ t < \tau(\omega, x))$ $\mathcal{F}_t$ on $\{\omega : t < \tau(\omega)\}$,

(★) holds a.s., $\forall t$.

If $(X_t^{(i)},\ t < \tau^{(i)})$, $i = 1,2$ a.s. are 2 (maximal) solutions with initial value X
Then $\tau^{(1)} = \tau^{(2)}$ a.s. and $X_t^{(1)} = X_t^{(2)}$.

If furthermore $\sigma_n, b \sim O(|x|)$, $n \to \infty$ i.e.

$\exists\ C$ st $\sum_1^M |\sigma_n(x)| + |b(x)| \le C(1 + |x|)$

Recall Thm 1: the solution X_t is $(\mathcal{F}_t^B)_t$ - adapted.

$X_t = \frac{1}{B_t(\omega) - 2}$ $\overset{Ito}{=}$ $\frac{1}{-2} + \int_0^t -X_s^2\, dB_s + \frac{1}{2}\int 2(X_s)^3\, ds$

↑ explosion $\tau = \inf\{B \ge [2, \infty)\}$.

$\sigma(B_s : 0 \le s \le t\}$ $\quad [0,t] \to \mathbb{R}$, $s \mapsto B_s(\omega)$

$\exists\ X_t(\omega) = F_t(X_0, B)$ ← fn on Wiener space $\qquad F$ is prog. meas $(W^d, \mathcal{B}W^d, \mathcal{B}\mathbb{R}^d$

$(t, x) \longmapsto F_t(x, B)$

Q how universal is $F_t(x, \cdot)$?
(1) If Y_t is a solution, is it true "$Y_t = F_t(Y_0, B)$" ?
(2) (B_t') ~ BM on $(\Omega', \mathcal{F}', \mathcal{F}_t', \mathbb{P}')$
Is $F_t(Y, \underbrace{B_\cdot'}_{B_t'(\omega')})$ a solution? that is.

that is,

$$Y_t = Y_0 + \sum \int_0^t \sigma_k(Y_s)\, d(B^k)_s + \int_0^t b(Y_s)\, ds \quad \text{a.s.} \; ?$$

Yes for globally Lipschitz σ, b.

☆ doesn't define 'solution', doesn't let you change prob. space.

Def 1 An $\mathbb{R}^d$ valued stoch process $(X_t, t \geq 0)$ defined on a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ is a solution of the SDE

Eq(σ, b) $\quad dX_t = \sum \sigma_k(X_t)\, dB_t + b(X_t)\, dt$

If $\exists$ a filtration $\mathcal{F}_t$ s.t.

(1) X_t is $\mathcal{F}_t$ adapted

(2) $\exists$ an $(\mathcal{F}_t)$-BM, $B_t = (B_t^1, \dots, B_t^d)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with $B_0 = 0$ a.s.

(3) The following identity holds $\forall\, t \geq 0$

$$X_t = X_0 + \sum \int_0^t \sigma_k(X_s)\, dB_s^k + \int_0^t b(X_s)\, ds \quad \text{a.s.}$$

(In particular the integrals exist)

$\widehat{\mathbb{R}}^d = \mathbb{R}^d \cup \{\Delta\}$ (Δ = '∞') If $\sigma : \mathbb{R}_+ \to \mathbb{R}^d$ is ctsly defined up to τ, set $\sigma(t) = \Delta \quad \forall t \geq \tau$.

$\widetilde{W}^d = C([0,\infty), \widetilde{\mathbb{R}}^d)$

eg. $f(x) = \left(0, \frac{1}{x-2}\right)$ $\lim_{x \to 2} |f(x)| = \infty$ $f(x) = \Delta \quad \forall x \geq 2$

Def 1' A d-dim adapted process $(X_t, t \in I)$ on a prob. space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a solution to $Eq(\sigma, b)$ if

$\exists$ a $(\mathcal{F}_t)$-BM (B_t) st $B_0 = 0$ a.s. and

On $\{\omega : t < \tau(\omega)\}$,

$$X_t = X_0 + \sum_{k=1}^{d} \int_0^t \sigma_k(X_r)\, dB_r^k + \int_0^t b(X_r)\, dr \quad \text{(a.s.)}$$

'ODE like concept'

Def 2 We say pathwise uniqueness holds for $Eq(\sigma, b)$ if whenever $(X_t), (X_t')$ are 2 solutions on the same prob. space, with the same BM and with $X_0 = X_0'$ a.s., then $X_t = X_t'$ $\forall t$ a.s. ('indistinguishable').

ex. 2a Tanaka Eq: 1dim, $dx_t = \operatorname{sgn} x_t \, dB_t$ on $\mathbb{R}$. $\operatorname{sgn} t = \begin{cases} -1 & x \leq 0 \\ +1 & x > 0 \end{cases}$

chose "=" st. left cts

$\Leftrightarrow$ x_t is a solution with driving BM, (B_t). $X_t = \cancel{X_0}^{\,0} + \int_0^t \operatorname{sgn} X_s \, dB_s$

with initial value 0.

Then $-x_t = \int_0^t -\operatorname{sgn} x_s \, dB_s = \int_0^t \operatorname{sgn}(-x_s)\, dB_s$ (except at $x=0$, 'lim in $\mathbb{P}$')

'B_s doesn't spend time at 0'

Tanaka eqn fails pathwise uniqueness.

Def 3 We say uniqueness in law holds if whenever (X_t) & (X_t') are two solutions of $Eq(\sigma, L)$ with $X_0 \overset{L}{=} X_0'$, then

law on Wiener space $\to \mathcal{L}(X_\bullet) := (X_\bullet)_* \mathbb{P} = \mathcal{L}(X_\bullet') := (X_\bullet')_* \mathbb{P}$

In defns for '!' we may take $X_0 = x_0 \in \mathbb{R}^d$ i.e. non-random.

(like - Watanabe: Regular Cond Prob.)

eg 2b Tanaka redux Let X_t be a solution with $X_0 = 0$: $X_t = \int_0^t \operatorname{sgn} X_s \, dB_s$.

(X_t) is a local mgale, cts in t

$\langle X, X \rangle_t = \int_0^t (\operatorname{sgn} X_s)^2 \, d\langle B, B \rangle_s = t$

Lévy Char $\Rightarrow$ X_t is a BM with $X_0 = 0$.

$\Rightarrow$ Its law on W_0^d is the Wiener measure.

so we have '!' in law.

Defn 4 We say existence in law holds if for any prob. dist. μ on $\mathbb{R}^d$, $\exists$ a soln with (X_t) with $\mathcal{L}(X_0) = \mu$.

(STP '$\exists$' for δ) $\mathbb{P}^x$ on W^d soln for δ_x ($X_0 = x$ a.e.)

$$\mathbb{P}^\mu := \int_{\mathbb{R}^d} \mathbb{P}^z(A) \, \mu(dx)$$ 'convolution!'

Tanaka (2c) We prove for $x = 0$ $\exists$ a solution to Tanaka's eq.

Let (W_t) be a BM on a (filtered) prob space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Define

$$\widetilde{B}_t = \int_0^t \operatorname{sgn}(W_s) \, dW_s, \quad \text{then } (\widetilde{B}_t) \text{ is a BM.}$$

Then (W_t) is a solution to Tanaka's eqn with driving BM, $(\widetilde{B}_t)$.

$$\int_0^t \operatorname{sgn} W_s \, d\widetilde{B}_s = \int_0^t \operatorname{sgn} W_s \operatorname{sgn} W_s \, dW_s = W_t$$

Rem (W_t) is NOT adapted to $\widetilde{B}_t$.

Defn A solution (X_t) with driving BM (B_t) is a *strong solution* if X_t is $\mathcal{F}_t^B$ adapted. Otherwise it is a *weak solution*

Sheet 8: Ito (move me to solutions notebook)

(43) $B_t = (B_t^1, \dots, B_t^n)$ is a BM

Claim: $|B_t|^2 = 2\sum_1^n \int_0^t B_s^i \, dB_s^i + nt$

Pf multi-dim Ito

$f(x) = |x|^2, \quad \partial_i f = 2x_i$

$\partial_{ij}^2 f(x) = 2\delta_{ij}$

Ito $\Rightarrow$ $f(B_t) = f(B_0) + \sum_{i=1}^n \int_0^t \partial_i f(B_s)\, dB_s^i + \frac{1}{2}\sum_{i,j}^n \int_0^t f(B_s)\, d\langle B^i, B^j\rangle_s.$

$= 0 + 2\sum_1^n \int_0^t B_s^i \, dB_s^i + \underbrace{\sum_{i=1}^n \int_0^t ds}_{=nt}$ □

Ex 44 (1) $B_t = (B_t^1, \dots, B_t^n)$, $n>1$ is a BM

define $r_t := |B_t|$

Q. An SDE for r_t?

Sol $f(x) = |x|, \ x \in \mathbb{R}^n$.

$\partial_i f(x) = \frac{x_i}{|x|}, \quad x \neq 0 \qquad \partial_{ii} f(x) = \frac{|x|^2 - x_i^2}{|x|^3} \quad x \neq 0.$

$T = \inf\{t : r_t = 0\}$. Consider $t < T$.

Ito: $df(B_t) = \sum_1^n \partial_i f(B_t)\, dB_t^i + \frac{1}{2}\sum_1^n \partial_{ii}^2 f(B_t)\, dt$

$\Leftrightarrow \quad dr_t = \underbrace{\sum_{i=1}^n \frac{B_t^i}{|B_t|} dB_t^i}_{\widetilde{B}_t} + \frac{1}{2}\sum_{i=1}^n \frac{|B_t|^2 - (B_t^i)^2}{|B_t|^3} dt$

$= \widetilde{B}_t + \frac{n-1}{|B_t|} dt = \widetilde{B}_t + \frac{n-1}{r_t} dt$

($\widetilde{B}$ is BM) $\widetilde{B}_t$ is a stoch integral : ct mgale, $\widetilde{B}_0 = 0$, $d\langle \widetilde{B}, \widetilde{B}\rangle_t = \sum \frac{(B_t^i)^2}{|B_t|^2} dt = dt$. $\Rightarrow$ (Levy) $\widetilde{B}$ is BM.

(2) given $g: \mathbb{R} \to \mathbb{R}$ bdd Borel meas.

WTS: $X_t = x_0 + \int_0^t X_s\, g(B_s)\, dB_s$.

Sol Define $M_t = \int_0^t g(B_s)\, dB_s$.

Then $\boxed{dX_t = X_t\, dM_t}$

guess $X_t = f(M_t + A_t)$, $+x_0$? $A_t \in BV$.

$$dX_t \overset{\text{ITÔ}}{=} \boxed{f'(M_t + A_t)(dM_t + dA_t) + \tfrac{1}{2} f''(M_t + A_t)\, d\langle M, M\rangle_t}.$$

① & ②: $X_t = f(M_t + A_t) = f'(M_t + A_t)$ — (A)

$0 = f'(M_t + A_t)\, dA_t + \tfrac{1}{2} f''(M_t + A_t)\, d\langle M\rangle_t$. — (B)

(A): Take $f(x) = Ce^x$

(B): $dA_t = -\tfrac{1}{2} d\langle M\rangle_t = -\tfrac{1}{2} g(B_t)^2$

$A_t = A_0 + \tfrac{1}{2}\int_0^t g(B_s)^2\, ds$.

$$X_t = x_0 + \tilde{C} \exp\left(\int_0^t g(B_s)\, dB_s - \tfrac{1}{2}\int_0^2 g(B_s)^2\, ds\right)$$

is a solution $\forall\, \tilde{C}$ □

$\boxed{dX_t = X_t\, dt \Rightarrow X_t = Ce^t}$

$X_t = C_t e^{M_t}$ guess

(45) • (N_t) a cts local mgl

• $N_0 = 0$

claim $X_t := e^{N_t - \frac{1}{2}\langle N,N\rangle_t} = e^{M_t}$ is a loc. mgl & $\mathbb{E} X_t \le 1$

$N_t - \tfrac{1}{2}\langle N, N\rangle_t$ is a semi-mgl

$f(x) = e^x$, Itô:

$X_t = 1 + \int_0^t f'(M_s)\, dM_s + \tfrac{1}{2}\int_0^t f''(M_s)\, d\langle M_s\rangle$

$= dN_s - \tfrac{1}{2} d\langle N\rangle_s$

$= 1 + \int_0^t X_s\, dN_s \Longrightarrow X_t$ is a loc. mgl

↙ T_n reduces X_t

$$\mathbb{E}X_t = \mathbb{E}\lim_{n\to\infty} X_n^{T_n} \overset{\text{Fatou}}{\leq} \lim_{n\to\infty} \mathbb{E}X_t^{T_n} = \lim_{n\to\infty} \mathbb{E}X_0 = 1 \qquad \square$$

(46) Show that a positive cts loc. mgl (N_t) with $N_0 = 1$ can be written as $N_t = e^{M_t - \frac{1}{2}\langle M\rangle_t}$, M is a Cts loc. mgl.

Pf Set $M_t = \int_0^t N_s^{-1}\, dN_s$

Itô $\Rightarrow$ $f(x) = \log(x)$, $x > 0$.

$$\log N_t = \log N_0 + \int_0^t \frac{dN_s}{N_s} + \frac{1}{2}\int_0^t \frac{d\langle N\rangle_t}{N_s^2} = 0 + M_t - \frac{1}{2}\langle M\rangle_t \qquad \square$$

(L) Missed 1 lecture.

Eq $E(\sigma, b)$, $dx_t = \sum_{k=1}^m \sigma_k(x_t)\, dB_t^k + b(X_t)\, dt$ on $\mathbb{R}^d$

let $x_t = (x_t^1 \dots x_t^d)$

$$\sigma_k = \begin{bmatrix} \sigma_k^1 \\ \vdots \\ \sigma_k^d \end{bmatrix} \qquad b = \begin{pmatrix} b^1 \\ \vdots \\ b^d \end{pmatrix}$$

$$\sigma = \begin{bmatrix} \sigma_1^1 & \cdots & \sigma_m^1 \\ \vdots & \ddots & \vdots \\ \sigma_1^d & \cdots & \sigma_m^d \end{bmatrix}$$

$$dx_t^i = \sigma_k^i(x_t)\, dB_t^k + b^i(x_t)\, dt$$

$$B_t = \begin{pmatrix} B_t^1 \\ \vdots \\ B_t^d \end{pmatrix} \qquad \begin{matrix} [\ ]\,[\,] & [\ ] \\ dx_t = \sigma(x_t)\, dB_t + b(x_t)\, dt \end{matrix}$$

ex $f: \mathbb{R}^d \to \mathbb{R}$ C^2 fn, x_t solves $Eq(\sigma, b)$

show:

$$f(x_t) = f(x_0) + \sum_{k=1}^m \sum_{j=1}^d \int_0^t \frac{\partial f}{\partial x_j}(x_s)\, \sigma_k^j(x_s)\, dB_s^k + \int_0^t \mathcal{L} f(x_s)\, ds$$

where $\mathcal{L} f(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_{j=1}^d b^j(x) \frac{\partial}{\partial x_j} f(x)$

Lem (1) If $f \in C^2(\mathbb{R}^d; \mathbb{R})$, x_t solve the above, then

$M_t^f := f(x_t) - f(x_0) - \int_0^t \mathcal{L} f(x_s)\, ds$ is a local martingale (Stoch integral √√)

(2) Let $W_t = ev_t$ be the canonical process on $(W^d, B(W^d), (B_t(W^d)), \hat{\mathbb{P}}_x)$

where $\hat{\mathbb{P}}_x = (X_\cdot)_* \mathbb{P}$, $X_\cdot$ solving the above SDE.

Then $\forall\ f \in C^2(\mathbb{R}^d, \mathbb{R})$, $f(W_t) - f(W_0) - \int_0^t \mathcal{L} f(W_s)\, ds$

is a local martingale

(1) $\iff$ (2) as we are talking about laws

Defn 4 A prob. measure μ ($= \mathbb{P}_x$ say) on the canonical space is said to solve the martingale problem for $\mathcal{L}$ if

$\forall\ f \in C_k^2(\mathbb{R}^d, \mathbb{R})$, $f(W_t) - f(W_0) - \int_0^t \mathcal{L} f(W_s)\, ds$ is a local martingale (wrt μ) on $(W^d, B(W^d), (B_t(W^d)), \mu)$

C_k^2 = cpt supp, ok as everything is local

Thm 3 (Page 169 I&W) The existence of solutions to $Eq(\sigma, b)$ is equivalent to the existence of a solution to the martingale problem

If $M_t^f := f(W_t) - f(W_0) - \int_0^t \mathcal{L} f(W_s)\, ds$ is loc. mgl then μ will solve $Eq(\sigma, b)$

Idea of Proof. $f_i(x) = x^i$, compute $M_t^{f_i}$. Integral Representation Thm for mgls.

every loc mgl, cts is a stoch integral

$\langle M^{f_i}, M^{f_j} \rangle_t$ is candidate for integrand (have a_{ij}') wrt a BM.

from ex: bracket is $\sum_k \partial_{x_i} \partial_{x_j} \sigma_j^k \sigma_i^k$, for this f_i, only 1 deriv does not vanish

find $\langle M^{f_i}, M^{f_j} \rangle_t$ using $f_{i,j}(x) = x_i x_j$.

Thm If $\forall x \in \mathbb{R}^d$, $\exists$ a unique solution $\mathbb{P}^x$ to the mgl problem assoc. to $\mathcal{L}$, & $\forall$ Borel set A of $\mathbb{R}^d$,

$$x \longmapsto \mathbb{P}^x(W_t \in A) \quad \text{is measurable}$$

Then $(W_t, \mathbb{P}^{x\cdot})$ is a Markov process with transition probabilities $P_t(x)$,

(W_t: canonical for)

where $P_t(x, A) = \mathbb{P}^x(W_t \in A)$. $= \mathbb{P}^x[\text{paths } w : W_t(w) \in A]$

Define $P_t f(x) := \int_{\mathbb{R}^d} f(y)\, P_t(x, dy)$ when f bdd, meas.

meas $\Rightarrow$ $P_t f$ is meas

$= \mathbb{E} f(F_t(x))$, $F_t(x)$ is a solution to $Eq(\sigma, b)$ w/ initial value x.

$P_t : \{\text{Bdd Meas fns}\} \circlearrowleft$ is called the Markov semigroup associated to $\mathcal{L}$.

Properties: $P_0 = id$

$P_s P_t = P_{s+t}$ (semigp property)

$\mathcal{L}$ is generator

$f \in D(\mathcal{L})$ if $\lim_{\varepsilon \to 0} \frac{P_\varepsilon f(x) - f(x)}{\varepsilon}$ exists. (call it $\mathcal{L}f$) If f is C_K^∞ say, $\sigma_{i,j}$ & b are cts.

$\lim_{\varepsilon \to 0} \frac{P_\varepsilon f(x) - f(x)}{\varepsilon} = \mathcal{L} f(x)$.

$\frac{d}{dt}\Big|_{t=0}^{h} P_t f(x)$

$$\lim_{\varepsilon \to 0} \frac{P_{t+\varepsilon} f - P_t f}{\varepsilon} = \frac{d}{dt} P_t f = \lim_{\varepsilon \to 0} P_t \left[\frac{P_\varepsilon f(x) - f(x)}{\varepsilon} \right] \overset{\text{"}}{=} P_t \mathcal{L} f$$

(uniformly something)

$$\frac{d}{dt} P_t f = P_t \mathcal{L} f$$

eg. heat if $\mathcal{L} = \Delta$

ex Let (u_t) be a solution to the eqn $\begin{cases} \frac{\partial u_t}{\partial t} = \mathcal{L} u_t & \forall t \quad (u_t) \text{ is } C^2 \text{ in } x, C^1 \text{ in } t \\ u_0 = f \end{cases}$

Then $u_t(x) = \mathbb{E} f(F_t(x))$ where $(F_t(x))$ are solution to Eq(σ, b) with initial value x.

$= \int f(y) P_t(x, dy) \overset{\mathcal{L} = \Delta}{=\!=} \int f(y) P_t(x, y) dy$

If elliptic then $P_t(x, dy) \ll dy$ & smooth density.

Last L

Thm 5 ① if (σ_k^i) & (b^i) are cts (& bdd), then for every x $\exists$ a solution to mgl problem.

② if $\sum_{i,j} a_{ij}\xi_i\xi_j \geq \theta|\xi|^2$, for all $\xi = (\xi_1 \cdots \xi_d) \in \mathbb{R}^d$ for some $\theta > 0$ & (a_{ij}), (b_i) are cts & bdd, then uniqueness in law holds.

Rem If σ_k, b are lipschitz cts, we have existence & uniqueness, smooth flow, (cts in Ic) cocycle property, strong markov property.

Rem ① $dx_t = dB_t$ has a unique solution for each x, $F_t(x) = x + B_t$.

$= \sum_{u=1}^{d} dB_t^u = \sum_1^d dB_t^k e_k$

② $\exists$ $\gamma : \mathbb{R}^d \to \mathbb{R}$ is cts, $\gamma \geq c > 0$. Then $dx_t = \gamma(x_t)\, dB_t$ is essentially $dx_t = dB_t$ by time change $A_t = \int_0^t \gamma(x_s)\, ds$.

③ Remove or add a drift: '$dx_t = \sigma(x_t)\, dB_t$' is essentially / can be compared to '$dx_t = \sigma(x_t)\, dB_t + \sigma(x_t) b(x_t)\, dt$'

eg. $dx_t = dB_t$ $\rightsquigarrow$ $dx_t = dB_t + b(x_t)\, dt$.

$\sigma\, dB_t$ $\qquad$ $\sigma\, dB_t + \sigma(\sigma^{-1}b \cdots$

Girsanov Transform: change of measure.

let $\mathbb{P} \underset{\text{equiv}}{\simeq} \mathbb{Q}$ be probability measures on $(\Omega, \mathcal{F}, \mathcal{F}_t)$. Let $f = \frac{d\mathbb{Q}}{d\mathbb{P}}$. $\exists$ $f \in L^1(\mathbb{P})$.

let $f_t = \mathbb{E}_{\mathbb{P}}[f \mid \mathcal{F}_t]$. Then (f_t) is a $\mathbb{P}$-mgl. Also, f_t is a strictly positive mgl.

exercise from before: ① $\exists$ a cts loc. mgl $\begin{cases} N_0 = 0 \\ (N_t) \end{cases}$ st $f_t = f_0 e^{N_t - \frac{1}{2}\langle N\rangle_t}$ (Itô)

HW ② given a semi-mgl, (M_t), $\langle f, M\rangle_t = ?$

$df_t = f_t\, dN_t$ ← is integral $\langle f, M\rangle_t = \langle \int_0^t f_s\, dN_s, dM_s\rangle = \int_0^t f_s\, d\langle M, N\rangle_s$.

In particular $\int_0^t \frac{1}{f_s}\, d\langle f, M\rangle_s = \int_0^t \frac{1}{f_s} f_s\, d\langle M, N\rangle_s = \langle M, N\rangle_t$

Thm (Girsanov Transform) Let (M_t) be a cts mgl on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.
Let $\widetilde{M}_t = M_t - \langle M, N\rangle_t$, where (N_t) is a cts loc. mgl.
Define $\mathbb{Q}$ by setting $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = f_t$.

Then $\widetilde{M}_t$ is a $\mathbb{Q}$-mgl.
We say $(\widetilde{M}_t)$ is the Girsanov Transform of (M_t).

(usually $f_0 \equiv 1$)

Applications. Let (B_t) (1D) $\sim BM(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Let $\mathbb{Q}$ be given by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = f_t = \exp\left(N_t - \tfrac{1}{2}\langle N\rangle_t\right) \quad \text{Let } \widetilde{B}_t = B_t - \langle B, N\rangle_t$$

Then $(\widetilde{B}_t)$ is a $\mathbb{Q}$-BM.

Let us consider $dY_t = dB_t$. Then $Y_t = Y_0 + B_t$, with initial value Y_0.

Let $b : \mathbb{R} \to \mathbb{R}$ be bdd & meas. Let $\widetilde{B}_t = B_t - \int_0^t b(Y_0 + B_s)\, ds$.
$\longrightarrow dY_t = d\widetilde{B}_t + b(Y_0 + B_s)\, ds$.
$\Longrightarrow Y_0 + B_t$ with driving noise $\widetilde{B}_t$, $(Y_0 + B_s, \widetilde{B}_t)$ solves $dx_t = dB_t^i + b(x_t)\,dt$.
Let

$$N_t = \int_0^t b(Y_s)\, dB_s; \quad \text{then} \quad \langle N, B\rangle_t = \int_0^t b(Y_s)\, ds \qquad \text{(the term removed above is } \langle N, B\rangle\text{)}$$

Define $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(N_t - \frac{1}{2}\langle N\rangle_t\right)$. Then $\widetilde{B}$ is a $\mathbb{Q}$-BM.

& $(Y_0 + B_t, \widetilde{B}_t)$ solve $dx_t = d\widetilde{B}_t + b(x_t)\,dt$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$.
Let f be Bdl & meas.

$$\mathbb{E}^{\mathbb{P}} f(X_t) \overset{\text{solution on } (\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})}{=} \int f(Y_0 + B_t)\, d\mathbb{Q} \quad \langle !\text{ in law}\rangle$$
$$= \int f(Y_0 + B_t) \exp\left[\int_0^t b(Y_0 + B_s)\, dB_s - \frac{1}{2}\int_0^t |b(Y_0 + B_s)|^2 ds.\right] d\mathbb{P}$$
initial x_0

Pf in notes

exam: H/W (+) remember Thms

notes: appendix ⇒ not from lectures.